J. Evol. Equ. 14 (2014), 913–947

1424-3199/14/040913-35, *published online* July 11, 2014
DOI 10.1007/s00028-014-0244-4

Journal of Evolution Equations

Existence and large-time asymptotics for solutions of semilinear parabolic systems with measure data

TOMASZ KLIMSIAK

Abstract. We study the Cauchy–Dirichlet problem for monotone semilinear uniformly elliptic second-order parabolic systems in divergence form with measure data. We show that under mild integrability conditions on the data, there exists a unique probabilistic solution of the system. We also show that if the operator and the data do not depend on time, then the solution of the parabolic system converges as $t \to \infty$ to the solution of the Dirichlet problem for an associated elliptic system. In fact, we prove some results on the rate of the convergence.

1. Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$ be an open bounded domain. In the present paper, we study systems of the form

$$\begin{cases} \frac{\partial u^k}{\partial t} - A_t u^k = f^k(t, x, u) + \mu^k \quad \text{in } D_T, \quad k = 1, \dots, N, \\ u_{|\partial D}(t, \cdot) = 0, \; t \in (0, T], \quad u(0, \cdot) = \varphi \quad \text{on } D. \end{cases} \tag{1.1}$$

Here, $D_T \equiv [0, T] \times D$, μ^k, $k = 1, \dots, N$ are bounded soft measures on $\mathbb{R}_+ \times D$, i.e., bounded Borel measures absolutely continuous with respect to the parabolic capacity determined by the operator

$$A_t = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(a_{ij}(t, x) \frac{\partial}{\partial x_i} \right). \tag{1.2}$$

In this paper, we assume that $f : D_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous monotone vector field, i.e., $f(t, x, \cdot)$ is continuous for a.e. $(t, x) \in D_T$ and for some $\alpha \in \mathbb{R}$,

$$\langle f(t, x, y) - f(t, x, y'), y - y' \rangle \leq \alpha |y - y'|^2 \tag{1.3}$$

for a.e. $(t, x) \in D_T$ and every $y, y' \in \mathbb{R}^N$. Concerning the growth of f, we merely require f to satisfy the following condition

$$f(\cdot, \cdot, 0) \in L^1(D_T), \quad \forall_{r>0,\, y \in \mathbb{R}^N} \quad R^{0,T}(\sup_{|y| \leq r} |f(\cdot, \cdot, y)|) < \infty, \quad m_1\text{-a.s.}, \tag{1.4}$$

Mathematics Subject Classification (2010): Primary 35J91, 35J57; Secondary 60H30
Keywords: Semilinear parabolic system, Divergence form operator, Measure data, Large-time behavior, Backward stochastic differential equation.

where $R^{0,T}$ is the potential operator of $\frac{\partial}{\partial t} - A_t$ on D_T. As for A_t, we assume that its coefficient $a : D_T \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq \sum_{ij=1}^{d} a_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d \tag{1.5}$$

for a.e. $(t, x) \in D_T$.

The aim of the present paper was twofold. In the first part of the paper, we focus on the problem of existence of solutions to (1.1). One of the possible approaches to the problem (1.1) is to use the framework of renormalized (or entropy) solutions. Such a framework was successfully applied to scalar equations of the type (1.1) with Leray-Lions-type operator A_t (see, e.g., [2]). For a function u on D_T, put

$$f_u(t,x) = f(t,x,u(t,x)), \quad (t,x) \in D_T.$$

We say that $u : D_T \to \mathbb{R}^k$ is a renormalized solution of (1.1) if $f_u \in L^1(D_T)$, u belongs to the space $\mathcal{T}_2^{0,1}$, i.e., for every $s \geq 0$, $T_s(u^k) = ((-s)\vee u^k)\wedge s \in W_2^{0,1}(D_T)$ for $k = 1, \dots, N$, and if u^k, $k = 1, \dots, N$ is a renormalized solution of the linear Cauchy–Dirichlet problem

$$\frac{\partial u^k}{\partial t} - A_t u^k = f_u^k(t,x) + \mu^k, \quad u^k(t,\cdot)_{|\partial D} = 0, \quad t \in (0,T], \quad u(0,\cdot) = \varphi.$$

Unfortunately, showing that $f_u \in L^1(D_T)$ under the growth condition (1.4) is in general a complicated, if ever possible, task. Consequently, we do not know whether $u \in \mathcal{T}_2^{0,1}$, and the concept of renormalized solutions is not applicable. However, we are able to prove that under (1.4), the function f_u is quasi-integrable, i.e., roughly speaking, integrable except possibly on sets of small capacity (see Sect. 5 for details). This in turn implies that u belongs to the stochastic Sobolev space $W^{0,1}(\mathbb{X}^{D_T})$ introduced in [14] to investigate the obstacle problem with possible nowhere Radon reflection measure. The space $W^{0,1}(\mathbb{X}^{D_T})$ is wider than $\mathcal{T}_2^{0,1}$ but their elements are regular enough to define correctly probabilistic solutions of (1.1) in $W^{0,1}(\mathbb{X}^{D_T})$ and prove the uniqueness result.

For $u \in \mathcal{B}(D_T)$, let $\bar{u}(t,x) = u(T-t,x)$, and for a bounded soft measure μ on D_T, let $\bar{\mu}$ be the bounded soft measure on D_T such that $\int \bar{\eta}\,\mathrm{d}\mu = \int \eta\,\mathrm{d}\bar{\mu}$ for every $\eta \in \mathcal{B}_b(D_T)$. Roughly speaking, $u \in W^{0,1}(\mathbb{X}^{D_T})$ is a probabilistic solution of (1.1) if for quasi-every (q.e. for short) $(s,x) \in D_T$,

$$\begin{aligned}\bar{u}(t,X_t) = \mathbf{1}_{\{\zeta^s > T\}}\varphi(X_T) + \int_t^{T\wedge\zeta^s} \bar{f}_u(r,X_r)\,\mathrm{d}r + \int_t^{T\wedge\zeta^s} \mathrm{d}A_r^{\bar{\mu}} \\ - \int_t^{T\wedge\zeta^s} \bar{\sigma}\nabla_{\mathbb{X}}\bar{u}(r,X_r)\,\mathrm{d}B_r, \quad t \in [0, T\wedge\zeta^s], \quad P_{s,x}\text{-a.s.},\end{aligned} \tag{1.6}$$

where $\sigma \cdot \sigma^T = a$, $\mathbb{X} = \{(X, P_{s,x}); (s,x) \in \mathbb{R}_+ \times D\}$ is a time-inhomogeneous Markov process associated with the operator A_t, ζ^s is the first exit time of $(X, P_{s,x})$ from D, i.e.,

$$\zeta^s = \inf\{t \geq s : X_t \notin D\}, \tag{1.7}$$

and A^μ is the additive functional of $\mathbb{X}$ in the Revuz correspondence with μ. In the last integral in (1.6), B is a standard d-dimensional Brownian motion and $\nabla_{\mathbb{X}}\bar{u}$ stands for the stochastic gradient of $\bar{u}$ (see [14] or Sect. 5). Formula (1.6) can be regarded as a nonlinear Feynman-Kac formula, because taking $t = s$ and integrating it with respect to the measure $P_{s,x}$, we get

$$\bar{u}(s,x) = E_{s,x}\left(\mathbf{1}_{\{\zeta^s > T\}}\varphi(X_T) + \int_s^{T\wedge\zeta^s} \bar{f}_u(r, X_r)\,\mathrm{d}r + \int_s^{T\wedge\zeta^s} \mathrm{d}A_r^{\bar{\mu}}\right) \tag{1.8}$$

whenever the above integrals exist. We would like to stress that our probabilistic solution u to (1.1) may be considered as some generalization of the notion of renormalized (or entropy) solution, because if $f_u \in L^1(D_T)$, then $u \in \mathcal{T}_2^{0,1}$, $u \in L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$ and u is a renormalized (and entropy as well) solution to (1.1) (see Remark 5.14). Perhaps, also the following comment is appropriate at this point, although the probabilistic solution of (1.1) is in general weak and at first glance its definition seems complicated, it is actually very convenient to deal with.

Our results on the existence and uniqueness of solutions of (1.1) generalize known results in the sense that we consider semilinear parabolic systems with measure data (semilinear elliptic systems with measure data are considered in [15,23]). We should also stress that our results are proved for systems with f satisfying quite general condition (1.3) for which the usual monotonicity methods do not apply and we only require f to satisfy mild integrability condition (1.4) analogous to the integrability condition considered for elliptic equations or systems in [2,15,23]. We also allow f to depend on x.

In the second part of the paper, we investigate the asymptotic behavior as $t \to \infty$ of probabilistic solutions of (1.1) in the case where $A_t = A$, f and $\mu = (\mu^1, \ldots, \mu^N)$ do not depend on time. Let $\tilde{\mu}^k(B) = \mu^k([0,1] \times B)$, $k = 1, \ldots, N$, for any Borel subset B of D. Our main result says that under the assumptions ensuring the existence and uniqueness of a solution v to the elliptic system

$$\begin{cases} -Av^k = f^k(x, v) + \tilde{\mu}^k & \text{in } D, \quad k = 1, \ldots, N, \\ v_{|\partial D} = 0 \end{cases} \tag{1.9}$$

we have

$$u(t,x) \to v(x) \quad \text{as} \quad t \to \infty$$

for q.e. $x \in D$. As a matter of fact, we prove that there is c depending only on d, Λ such that for every $t > 0$ and $q \in (0,1)$,

$$|u(t,x)-v(x)| \le c(1-q)^{-1/q}\,t^{-d/2}\Big((P_x(\zeta^0>t))^{(1-q)/q}\|\varphi\|_{L^1} \\ +\|Rf(\cdot,0)\|_{L^1}+\|R\tilde{\mu}\|_{L^1} \le c(1-q)^{-1/q}\,t^{-d/2}\Big) \tag{1.10}$$

for q.e. $x\in D$, where R is the potential operator of $-A$ on D. From this, it follows in particular that there exists $c=c(\Lambda,d,|D|)$ such that for every $t>0$,

$$|u(t,x)-v(x)| \le c(1-q)^{-1/q}\,t^{-d/2}(\mathrm{e}^{-bt(1-q)/q}\|\varphi\|_{L^1}+\|f(\cdot,0)\|_{L^1}+\|\tilde{\mu}\|_{TV}), \tag{1.11}$$

because it is known that $P_x(\zeta^0>t)\le a\mathrm{e}^{-bt}$ for some $a,b>0$ depending only on d, Λ and $|D|$ (the Lebesgue measure of D) and $\|R\tilde{\mu}\|_{L^1}\le C(d,\lambda,|D|)\|\tilde{\mu}\|_{TV}$ ($\|\tilde{\mu}\|_{TV}$ stands for the total variation of $\tilde{\mu}$). For instance, if $f(\cdot,0)=0$ and $\mu(\mathrm{d}x)=0$, then the rate of convergence in (1.11) is the same as in the classical case of one linear equation (see [8]). We also show that in fact (1.10) holds for every x from the set

$$F_0=\{x\in D;\forall\, r>0\ \ R|f(\cdot,0)|(x)+(R\tilde{\mu})(x)+R(\sup_{|y|\le r}|f(\cdot,y)|)(x)<\infty\}. \tag{1.12}$$

In case $N=1$, the large-time asymptotic behavior of solutions to parabolic equations with measure or L^1 data was investigated in [21,27–29] (in [27], the case of general, possibly singular measures is considered). In all these papers in proofs, some comparison results are used. Therefore, the methods of [21,27–29] cannot be applied to systems considered in the present paper. Let us also point out that these methods do not provide estimates of the difference between solutions to parabolic equations and the corresponding stationary solutions.

Although the main results of the paper concern systems of PDEs and are analytic in nature, the methods of proofs are those of stochastic analysis, Markov processes, and especially the theory of backward stochastic differential equations. Therefore, in Sects. 2–4, we give relevant background material concerning these topics. Then, in Sect. 5, we prove our results on the existence and uniqueness of solutions of (1.1), and in Sect. 6 results on their large-time behavior. Our idea of using the methods of backward stochastic differential equations to the study of large-time behavior of semilinear parabolic equations is new. It seems likely that it can be applied to wider that (1.1) class of equations.

2. Preliminary results

Let us fix a probability space $(\Omega,\mathcal{F},P)$ equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions.

By B, we denote a standard d-dimensional $\{\mathcal{F}_t\}$-Brownian motion. By $\mathcal{A}$, we denote the set of all $\{\mathcal{F}_t\}$ progressively measurable real-valued processes and by $\mathcal{V}$ (respectively, $\mathcal{V}_c$) the subspace of $\mathcal{A}$ consisting of all increasing càdlàg (respectively, continuous) processes Y such that $Y_0=0$. M is the space of all processes $Z\in\mathcal{A}$ such that $P(\int_0^T|Z_t|^2\,\mathrm{d}t<\infty)=1$ for every $T>0$. M^p, $p>0$, is the subspace of

M consisting of all processes such that $E(\int_0^\infty |Z_r|^2\,\mathrm{d}r)^{p/2} < \infty$. By $\mathcal{D}$ (respectively, $\mathcal{S}$), we denote the space of all càdlàg (respectively, continuous) processes in $\mathcal{A}$, and by $\mathcal{D}^p$ (respectively, $\mathcal{S}^p$), $p > 0$, the space of all processes $Y \in \mathcal{D}$ (respectively, $Y \in \mathcal{S}$) such that $E\sup_{t\geq 0}|Y_t|^p < \infty$. We say that a process Y is of class (D) if $Y \in \mathcal{A}$ and the family $\{Y_\tau,\ \tau \in \mathcal{T}\}$, where $\mathcal{T}$ is the set of all finite $\{\mathcal{F}_t\}$-stopping times, is uniformly integrable. For a càdlàg process Y, we write

$$\Delta Y_t = Y_t - Y_{t-}, \quad Y_{t-} = \lim_{s\nearrow t} Y_s.$$

In the paper, we adopt the following convention. If S is a space of real functions and $N, d \in \mathbb{N}$, then by $[S]^N$ (respectively, $[S]^{N\times d}$), we denote the space of all functions of the form $f = (f^1,\dots,f^N)$ (respectively, $f = [f^{i,j}]_{N\times d}$) such that $f^i \in S$ for $i = 1,\dots,N$ (respectively, $f^{i,j} \in S$ for $i = 1,\dots,N,\ j = 1,\dots,d$). If A is an $N \times d$-dimensional real matrix, then $|A|$ stands for $\operatorname{trace} AA^*$.

Write

$$\hat{x} = \hat{\operatorname{sgn}}(x) = \mathbf{1}_{\{x\neq 0\}}\frac{x}{|x|}, \quad x \in \mathbb{R}^N.$$

The following multidimensional version of the Itô–Tanaka formula will be frequently used in the paper.

PROPOSITION 2.1. *Let X be a progressively measurable process such that*

$$X_t = X_0 + \int_0^t \mathrm{d}K_s + \int_0^t H_s\,\mathrm{d}B_s, \quad t \geq 0$$

for some $K \in [\mathcal{V}]^N$, $H \in [M]^{N\times d}$. Then, there is $L \in \mathcal{V}$ such that for every $p \geq 1$,

$$\begin{aligned}|X_t|^p &= |X_0|^p + p\int_0^t |X_{s-}|^{p-1}\langle \hat{X}_{s-}, \mathrm{d}K_s\rangle + p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, H_s\,\mathrm{d}B_s\rangle \\ &\quad + \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\{(2-p)(|H_s|^2 - \langle \hat{X}_s, H_sH_s^*\hat{X}_s\rangle) + (p-1)|H_s|^2\}\,\mathrm{d}s \\ &\quad + \mathbf{1}_{\{p=1\}}L_t + \sum_{0<s\leq t}(\Delta|X_s|^p - \langle |X_s|^{p-1}\hat{X}_s, \Delta X_s\rangle), \quad t \geq 0.\end{aligned}$$

Proof. The proof is a modification of the proof of [3, Lemma 2.2]. Obviously, it suffices to prove the formula for $t \in [0,T]$. Set $u_\varepsilon(x) = (|x|^2+\varepsilon^2)^{1/2}$, $x \in \mathbb{R}^N$, $\varepsilon > 0$. A straightforward computation shows that

$$\nabla u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)x,\ D^2u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)I + p(p-2)u_\varepsilon^{p-4}(x)(x\otimes x),$$

where I is the n-dimensional identity matrix. By the Itô–Meyer formula,

$$
\begin{aligned}
u^p_\varepsilon(X_s) &= u^p_\varepsilon(X_0) + \int_0^t \langle \nabla u^p_\varepsilon(X_{s-}), \mathrm{d}K_s\rangle + \int_0^t \langle \nabla u^p_\varepsilon(X_{s-}), H_s\,\mathrm{d}B_s\rangle \\
&\quad + \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 u^p_\varepsilon}{\partial x_i \partial x_j}(X_{s-}) d[X^i, X^j]^c_s + \sum_{0<s\le t}(\Delta u^p_\varepsilon(X_s) - \langle \nabla u^p_\varepsilon(X_{s-}), \Delta X_s\rangle) \\
&= u^p_\varepsilon(X_0) + \sum_{i=1}^4 I^\varepsilon_i(t).
\end{aligned}
$$

It is clear that

$$
u^p_\varepsilon(X_t) \to |X_t|^p, \quad I^\varepsilon_1(t) \to p\int_0^t |X_{s-}|^{p-1}\langle \hat{X}_{s-}, \mathrm{d}K_s\rangle, \quad \varepsilon \searrow 0
$$

a.s. for $t \in [0, T]$. Using arguments from the proof of [3, Lemma 2.2], one can show that

$$
I^\varepsilon_2(t) \to p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, H_s\,\mathrm{d}B_s\rangle, \quad \varepsilon \searrow 0
$$

uniformly on $[0, T]$ in probability, and that $I^\varepsilon_3(t) = I^\varepsilon_{3,1}(t) + I^\varepsilon_{3,2}(t)$, where

$$
\begin{aligned}
I^\varepsilon_{3,1}(t) &= \frac{p(p-2)}{2}\int_0^t \Big(|X_s| u^{-1}_\varepsilon(X_s)\Big)^{4-p} |X_s|^{p-2}\mathbf{1}_{\{X_s\neq 0\}}\Big(|H_s|^2 - \langle \hat{X}_s, H_s H^*_s \hat{X}_s\rangle\Big)\mathrm{d}s \\
&\quad + p(p-1)(|X_s| u^{-1}_\varepsilon(X_s))^{4-p}|X_s|^{p-2}\mathbf{1}_{\{X_s\neq 0\}}|H_s|^2
\end{aligned}
$$

and

$$
I^\varepsilon_{3,2}(t) = \varepsilon^2 \frac{p}{2}\int_0^t |H_s|^2 u^{p-4}_\varepsilon(X_s)\,\mathrm{d}s.
$$

Moreover, as $\varepsilon \searrow 0$,

$$
I^\varepsilon_{3,1}(t) \to \int_0^t |X_s|^{p-1}\mathbf{1}_{\{X_s\neq 0\}}\Big\{(2-p)\Big(|H_s|^2 - \langle \hat{X}_s, H_s H^*_s \hat{X}_s\rangle\Big) - (p-1)|H_s|^2\Big\}\,\mathrm{d}s
$$

a.s. for every $t \in [0, T]$. We now show the convergence of $I^\varepsilon_4(t)$. It is clear that

$$
\Delta u^p_\varepsilon(X_s) \to \Delta|X_s|^p, \quad \langle \nabla u^p_\varepsilon(X_{s-}), \Delta X_s\rangle \to \langle |X_s|^{p-1}\hat{X}_s, \Delta X_s\rangle, \quad s \in [0, T].
$$

Observe also that

$$
|\Delta u^p_\varepsilon(X_s)| \le \sup_{\theta\in[0,1]} |\nabla u^p_\varepsilon(\theta \Delta X_s + X_{s-})||\Delta X_s| \le 3\sup_{0\le t\le T}(|X_t|^{p-1} + 1)|\Delta X_s|
$$

and

$$
|\nabla u^p_\varepsilon(X_{s-})| \le \sup_{0\le t\le T}(|X_t|^{p-1} + 1).
$$

Since $\sum_{0<t<T} |\Delta X_s| \leq |K_T|$, applying the Lebesgue dominated convergence theorem shows that for every $t \in [0, T]$,

$$I_4^{\varepsilon}(t) \to \sum_{0<s\leq t} \Big(\Delta |X_s|^p - \langle |X_s|^{p-1} \hat{X}_s, \Delta X_s \rangle \Big), \quad \varepsilon \searrow 0.$$

By what has already been proved, it follows that $I_{3,2}^{\varepsilon}(t)$ is convergent. Put $L_t(p) = \lim_{\varepsilon \to 0} I_{3,2}^{\varepsilon}(t)$. Then, L is a càdlàg increasing process, and as in the proof [3, Lemma 2.2], one can show that if $p > 1$, then $L_t(p) = 0$ for $t \in [0, T]$, which completes the proof. $\square$

REMARK 2.2. It is well known that the function $v(x) = p|x|^{p-1}\hat{x}$ is the subgradient of the function $u(x) = |x|^p$. Therefore, $u(x) - u(y) \geq \langle x - y, v(x) \rangle,\ x, y \in \mathbb{R}^N$. This implies that the process

$$I_t = \sum_{0<s\leq t} \Big(\Delta |X_s|^p - \langle |X_s|^{p-1} \hat{X}_s, \Delta X_s \rangle \Big), \quad t \geq 0$$

is increasing.

COROLLARY 2.3. *Under the assumptions of Proposition 2.1, for every $0 \leq t \leq T$ and $p \geq 1$,*

$$|X_t|^p + c(p) \int_t^T |X_s|^{p-2} \mathbf{1}_{\{X_s \neq 0\}} |H_s|^p \, \mathrm{d}s \leq |X_T|^p - p \int_t^T |X_{s-}|^{p-1} \langle \hat{X}_{s-}, \mathrm{d}K_s \rangle - \int_t^T |X_s|^{p-1} \langle \hat{X}_s, H_s \, \mathrm{d}B_s \rangle,$$

where $c(p) = p[(p-1) \wedge 1]/2$.

3. Backward stochastic differential equations

Let B denote a standard d-dimensional $\{\mathcal{F}_t\}$-Brownian motion. Let σ be a bounded $\{\mathcal{F}_t\}$-stopping time, ξ be an $\mathcal{F}_\sigma$-measurable random variable, $A \in \mathcal{V}$ and let $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a measurable function such that $f(\cdot, y)$ is progressively measurable for every $y \in \mathbb{R}^N$.

Let us recall that a pair (Y, Z) consisting of an $\mathbb{R}^N$-valued process Y and an $\mathbb{R}^{N\times d}$-valued processes Z is called a solution of BSDE$(\xi, \sigma, f + \mathrm{d}A)$ if

(a) Y, Z are $\{\mathcal{F}_t\}$-progressively measurable, Y is càdlàg, $P(\int_0^\sigma |Z_t|^2 \, \mathrm{d}t < \infty) = 1$,
(b) $t \mapsto f(t, Y_t) \in L^1(0, \sigma),\ P$-a.s.,
(c) $Y_t = \xi + \int_t^\sigma f(s, Y_s) \, \mathrm{d}s + \int_t^\sigma \mathrm{d}A_s - \int_t^\sigma Z_s \, \mathrm{d}B_s,\ 0 \leq t \leq \sigma,\ P$-a.s.

Let (Y, Z) be a solution of BSDE$(\xi, \sigma, f + \mathrm{d}A)$. By putting $Y_t = \xi, t \geq \sigma,\ Z_t = 0,\ t \geq \sigma$, we may and will assume in the sequel that the processes Y, Z are defined for $t \geq 0$. We also adopt the convention that $\int_a^b = 0$ for $a \geq b$. Then, the stochastic equation in (c) is satisfied for every $t \geq 0$.

Let us consider the following assumptions.

(A1) $E\left(|\xi|^p+(\int_0^\sigma|f(t,0)|\,\mathrm{d}t)^p+|A|_\sigma^p\right)<+\infty$.
(A2) There is $\mu\in\mathbb{R}$ such that $\langle y-y', f(t,y)-f(t,y')\rangle\le\mu|y-y'|^2$ for every $t\ge 0,\ y,y'\in\mathbb{R}^N$.
(A3) For every $t\ge 0,\ y\mapsto f(t,y)$ is continuous.
(A4) For every $r>0,\ E\int_0^\sigma\sup_{|y|\le r}|f(t,y)|\,\mathrm{d}t<\infty$.

In [3, Theorem 4.2], it is proved that under (A1)–(A4) with $p>1$, there exists a unique solution $(Y,Z)\in\mathcal{S}^p\otimes M^p$ of BSDE(ξ,σ,f). We will show how to modify the proof of [3, Theorem 4.2] to get the existence and uniqueness in the general case, i.e., for $p\ge 1$ and nonzero process A.

The proof of [3, Theorem 4.2] is based on Lemma 3.1 and Proposition 3.2 in [3], [26, Theorem 2.2] and [4, Lemma 2.2]. To state the generalizations of Lemma 3.1 and Proposition 3.2 in [3] to equations with nonzero $\mathrm{d}A$, we need the following hypothesis.

(A) There is $\mu\in\mathbb{R}$ and a nonnegative progressively measurable process $\{f_t,\ t\ge 0\}$ such that $\langle\hat{y}, f(t,y)\rangle\le f_t+\mu|y|$ for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}^N$.

For a process $A\in\mathcal{V}$, we denote by $|A|_t$ its variation on the interval $[0,t]$.

LEMMA 3.1. *Assume (A). Let (Y,Z) be a solution of BSDE$(\xi,\sigma,f+\mathrm{d}A)$. If $Y\in\mathcal{D}^p$ and $E(\int_0^\sigma f_t\,\mathrm{d}t)^p+E|A|_\sigma^p<\infty$ for some $p>0$, then $Z\in M^p$ and there exists C_p depending only on p such that for every $a\ge\mu$,*

$$E\left(\int_0^\sigma e^{2at}|Z_t|^2\,\mathrm{d}t\right)^{p/2}\le C_pE\left(\sup_{0\le t\le\sigma}e^{apt}|Y_t|^p+\left(\int_0^\sigma e^{at}f_t\,\mathrm{d}t\right)^p+\left(\int_0^\sigma e^{at}\,\mathrm{d}|A|_t\right)^p\right).$$

Proof. The proof goes through as for [3, Lemma 3.1], with obvious changes. □

PROPOSITION 3.2. *Assume (A). Let (Y,Z) be a solution of BSDE$(\xi,\sigma,f+\mathrm{d}A)$. If $Y\in\mathcal{D}^p$ and $E(\int_0^\sigma f_t\,\mathrm{d}t)^p+E|A|_\sigma^p<\infty$ for some $p>1$, then there exists C_p depending only on p, such that for every $a\ge\mu$,*

$$E\left(\sup_{0\le t\le\sigma}e^{apt}|Y_t|^p+\left(\int_0^\sigma e^{2at}|Z_t|^2\,\mathrm{d}t\right)^{p/2}\right)\le C_pE\left(e^{ap\sigma}|\xi|^p+\left(\int_0^\sigma e^{at}f_t\,\mathrm{d}t\right)^p+\left(\int_0^\sigma e^{at}\mathrm{d}|A|_t\right)^p\right).$$

Proof. It suffices to repeat, with obvious changes, arguments from the proof of [3, Proposition 3.2]. The only difference is that we use our Corollary 2.3 instead of [3, Corollary 2.3]. □

We now prove the analogues of [26, Theorem 2.2] and [4, Lemma 2.2] for equations with nonzero $\mathrm{d}A$.

LEMMA 3.3. *Assume that*

$$\langle f(t,y),y\rangle\le c|y|^2,\quad y\in\mathbb{R}^N,\quad t\ge 0 \tag{3.1}$$

for some $c \geq 0$ *and* $\|\xi\|_\infty + c + \||A|_\sigma\|_\infty \leq r < \infty$. *If* (Y, Z) *is a solution of BSDE*$(\xi, \sigma, f + \mathrm{d}A)$ *such that* Y *is of class (D), then* $\|Y\|_\infty \leq r$.

Proof. By Corollary 2.3,

$$|Y_t| \leq |\xi| + \int_t^\sigma \langle f(s, Y_s), \hat{Y}_s \rangle \, \mathrm{d}s + \int_t^\sigma \langle \hat{Y}_s, \mathrm{d}A_s \rangle - \int_t^\sigma Z_s \, \mathrm{d}B_s, \quad t \geq 0.$$

Since Y is of class (D), it follows from (3.1) that $|Y_t| \leq E^{\mathcal{F}_t}(|\xi| + c + |A|_\sigma)$, $t \geq 0$, which implies the desired estimate. □

LEMMA 3.4. *Assume* (A1)–(A3) *are satisfied with* $p = 2$ *and that* $|f(t, y)| \leq c + |f(t, 0)|$ *for every* $t \geq 0$, $y \in \mathbb{R}^N$. *Then, there exists a unique solution* (Y, Z) *of BSDE*$(\xi, \sigma, f + \mathrm{d}A)$ *such that* $(Y, Z) \in \mathcal{D}^2 \otimes M^2$.

Proof. Using standard arguments, one can prove the existence of a unique solution $(\overline{Y}, \overline{Z}) \in \mathcal{D}^2 \otimes M^2$ of the BSDE

$$\overline{Y}_t = \int_t^\sigma \mathrm{d}A_s - \int_t^\sigma \overline{Z}_s \, \mathrm{d}B_s.$$

Furthermore, from [26], it follows that under the assumptions of the lemma, there exists a unique solution $(Y', Z') \in \mathcal{D}^2 \otimes M^2$ of the BSDE

$$Y'_t = \xi + \int_t^\sigma \bar{f}(s, Y'_s) \, \mathrm{d}s - \int_t^\sigma Z'_s \, \mathrm{d}B_s, \quad t \in [0, \sigma]$$

with $\bar{f}(t, y) = f(t, y + \overline{Y}_t)$. Set $(Y, Z) = (Y' + \overline{Y}, Z' + \overline{Z})$. Then, (Y, Z) is a solution of BSDE$(\xi, \sigma, f + \mathrm{d}A)$. □

We are now ready to formulate and prove the existence and uniqueness result in case $p > 1$.

THEOREM 3.5. *Let* $p > 1$. *If* (A1)–(A4) *are satisfied, then there exists a unique solution* $(Y, Z) \in \mathcal{D}^p \times M^p$ *of BSDE*$(\xi, \sigma, f + \mathrm{d}A)$.

Proof. Step 1. We first assume that $a \equiv \|f(\cdot, 0)\|_\infty + \|\xi\|_\infty + \||A|_\sigma\|_\infty < \infty$. Let us define h_n as in the first step of the proof of [3, Theorem 4.2] (with $r > a$). Then, in much, the same way as in the proof of that theorem, but using Lemma 3.1 and Proposition 3.2 instead of [3, Lemma 3.1] and [3, Proposition 3.2] and Lemmas 3.3 and 3.4 instead of [26, Theorem 2.2] and [4, Lemma 2.2], one can prove that there exists a solution $(Y, Z) \in \mathcal{D}^2 \otimes M^2$ of BSDE$(\xi, \sigma, f + \mathrm{d}A)$.

Step 2. We define ξ_n, f_n as in the second step of the proof of [3, Theorem 4.2] and set $A^n_t = \int_0^t \mathbf{1}_{\{|A|_s \leq n\}} \, \mathrm{d}A_s$. The proof of the existence of a solution $(Y, Z) \in \mathcal{D}^p \otimes M^p$ of BSDE$(\xi, \sigma, f + \mathrm{d}A)$ goes as the proof of [3, Theorem 4.2], the only difference being in the use of Lemma 3.1 and Proposition 3.2 instead of [3, Lemma 3.2] and [3, Proposition 3.2]. □

We now turn to the case $p = 1$. We first prove the uniqueness result.

THEOREM 3.6. *Let $p = 1$. If* (A2) *is satisfied, then there exists at most one solution (Y, Z) of BSDE$(\xi, \sigma, f + \mathrm{d}A)$ such that Y is of class (D).*

Proof Without the loss of generality, we may assume that $\mu \leq 0$. Let (Y, Z), (Y', Z') be the solutions of BSDE$(\xi, \sigma, f + \mathrm{d}A)$ such that Y, Y' are of class (D). Then, $(\overline{Y}, \overline{Z}) = (Y - Y', Z - Z')$ is a solution of the BSDE

$$\overline{Y}_t = \int_t^\sigma f(s, Y_s) - f(s, Y'_s)\,\mathrm{d}s - \int_t^\sigma \overline{Z}_s\, hB_s, \quad t \geq 0.$$

Let $\tau_k = \inf\{t \geq 0; \int_0^t |\overline{Z}_s|^2\,\mathrm{d}s \geq k\}$. By the Itô–Meyer formula and (A2),

$$|\overline{Y}_t| \leq |\overline{Y}_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} \langle f(s, Y_s) - f(s, Y'_s), \hat{\mathrm{sgn}}(\overline{Y}_s)\rangle\,\mathrm{d}s$$
$$- \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\overline{Y}_s), \overline{Z}_s\,\mathrm{d}B_s\rangle \leq - \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\overline{Y}_s), \overline{Z}_s\,\mathrm{d}B_s\rangle, \quad t \geq 0.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of the above inequality and then letting $k \to \infty$ and using the fact that $\overline{Y}$ is of class (D), we conclude that $|\overline{Y}_t| = 0,\ t \geq 0$. □

For $k > 0$, let us put

$$T_k(y) = \frac{ky}{k \vee |y|}, \quad y \in \mathbb{R}^N.$$

THEOREM 3.7. *Let $p = 1$. If* (A1)–(A4) *are satisfied, then there exists a solution (Y, Z) of BSDE$(\xi, \sigma, f + \mathrm{d}A)$ such that $(Y, Z) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and Y is of class (D).*

Proof. Without the loss of generality, we may assume that $\mu \leq 0$. Set

$$\xi^n = T_n(\xi), \quad f_n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0)), \quad A^n_t = \int_0^t \mathbf{1}_{\{|A|_s \leq n\}}\,\mathrm{d}A_s.$$

By Theorem 3.5, for every $n \in \mathbb{N}$, there exists a solution $(Y^n, Z^n) \in \mathcal{D}^2 \otimes M^2$ of BSDE$(\xi^n, \sigma, f_n + \mathrm{d}A^n)$. Let $m \geq n$. Write $\delta Y = Y^m - Y^n$, $\delta Z = Z^m - Z^n$, $\delta\xi = \xi^m - \xi^n$ and

$$\tau_k = \inf\left\{t \geq 0; \int_0^t |\delta Z_s|^2\,\mathrm{d}s > k\right\}.$$

By the Itô–Meyer formula, for $t \geq 0$, we have

$$|\delta Y_{t\wedge\tau_k}| \leq |\delta Y_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\delta Y_s), f_m(s, Y^m_s) - f_n(s, Y^n_s)\rangle\,\mathrm{d}s$$
$$+ \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\delta Y_{s-}), d(A^m_s - A^n_s)\rangle + \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\delta Y_s), \delta Z_s\,\mathrm{d}B_s\rangle$$
$$\leq |\delta Y_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} |f_m(s, Y^n_s) - f_n(s, Y^n_s)|\,\mathrm{d}s$$
$$+ \int_t^{\tau_k\wedge\sigma} \mathrm{d}|A^m - A^n|_s + \int_t^{\tau_k\wedge\sigma} \langle \hat{\mathrm{sgn}}(\delta Y_s), \delta Z_s\,\mathrm{d}B_s\rangle,$$

the last inequality being a consequence of monotonicity of f_n with respect to y. Conditioning with respect to $\mathcal{F}_t$ and using the fact that δY is of class (D), we conclude from the above inequality that

$$|\delta Y_t| \le E^{\mathcal{F}_t}\left(|\xi|\mathbf{1}_{\{|\xi|>n\}} + \int_0^\sigma |f(s,0)|\mathbf{1}_{\{|f(s,0)|>n\}}\,\mathrm{d}s + \int_0^\sigma \mathbf{1}_{\{|A|_s>n\}}\,\mathrm{d}|A|_s\right)$$

for $t \ge 0$. To complete the proof, it suffices now to repeat step by step the arguments following Eq. (12) in the proof of [3, Proposition 6.4]. □

4. Markov processes and potential theory

To make our exposition in the next sections self-contained, in this section, we recall some useful facts about diffusions associated with the operator A_t defined by (1.2), their additive functionals and the Revuz correspondence between these functionals and soft measures.

4.1. Time-inhomogeneous diffusions

Let $\Omega = C(\mathbb{R}_+;\mathbb{R}^d)$ be the space of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+ = [0,\infty)$, X be the canonical process on Ω, $\mathcal{F}^0_{s,t} = \sigma(X_u, u \in [s,t])$. We define $\mathcal{F}_{s,\infty}$ as the completion of $\mathcal{F}^0_{s,\infty}$ with respect to the family $\mathcal{P} = \{P_{s,\mu} : \mu$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot)\,\mu(\mathrm{d}x)$, and then we define $\mathcal{F}_{s,t}$ as the completion of $\mathcal{F}^0_{s,t}$ in $\mathcal{F}_{s,\infty}$ with respect to $\mathcal{P}$.

Let p denote the fundamental solution for the operator A_t defined by (1.2). It is known (see [31]) that there exists a unique time-inhomogeneous Markov process $\mathbb{X} = \{(X, P_{s,x}) : (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ associated with A_t. Namely, $\mathbb{X}$ is a unique Markov process for which p is the transition density function, i.e.,

$$P_{s,x}(X_t = x;\ 0 \le t \le s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_\Gamma p(s,x,t,y)\,dy, \quad t > s$$

for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. It is known (see [32]) that $\mathbb{X}$ admits the so-called strict Fukushima decomposition, i.e., for every $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$X_t - X_s = A_{s,t} + M_{s,t}, \quad s \le t, \quad P_{s,x}\text{-a.s.},$$

where M is a two-parameter martingale additive functional (MAF) of $\mathbb{X}$ of finite energy and A is a two-parameter continuous additive functional (CAF) of $\mathbb{X}$ of zero energy. Moreover,

$$\langle M^i_{s,\cdot}, M^j_{s,\cdot}\rangle_t = \int_s^t a_{ij}(r, X_r)\,\mathrm{d}r, \quad s \le t, \tag{4.1}$$

which implies in particular that the process

$$B_{s,t} = \int_s^t \sigma^{-1}(r, X_r)\,\mathrm{d}M_{s,r}, \quad t \ge s, \tag{4.2}$$

where $\sigma \cdot \sigma^T = a$ is a Brownian motion under $P_{s,x}$. It is also known (see [20]) that $B_{s,\cdot}$ is an $\{\mathcal{F}_{s,t}\}_{t\geq s}$-Brownian motion.

4.2. Time-homogeneous diffusions

In what follows we will also make substantial use of time-homogeneous Markov process $\mathbb{X}'$ associated with the operator $\frac{\partial}{\partial t} + A_t$ (see [24]). A brief sketch of a useful construction of $\mathbb{X}'$ on an extension of Ω is given below.

We set

$$\Omega' = \mathbb{R}_+ \times \Omega, \quad P'_{s,x}(B) = P_{s,x}(\{\omega \in \Omega : (s,\omega) \in B\}) \tag{4.3}$$

and consider the process $\mathbf{X}$ on Ω' defined as

$$\mathbf{X}_t(s,\omega) = (s+t, X_{s+t}(\omega)), \quad t \geq 0. \tag{4.4}$$

Let $\mathcal{F}_t'^0 = \sigma(\mathbf{X}_u, u \leq t)$, $\mathcal{F}_\infty'^0 = \sigma(\mathbf{X}_u, u < \infty)$ and let $\mathcal{F}'_\infty$ denote the completion of $\mathcal{F}_\infty'^0$ with respect to the family $\mathcal{P}' = \{P'_\mu : \mu$ is a probability measure on $\mathbb{R}_+ \times \mathbb{R}^d\}$ and $\mathcal{F}'_t$ denote the completion of $\mathcal{F}_t'^0$ in $\mathcal{F}'_\infty$ with respect to $\mathcal{P}'$. Then, $\mathbb{X}' = \{(\mathbf{X}_t, P'_{s,x}); (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a time-homogeneous Markov process with respect to the filtration $\{\mathcal{F}'_t\}$ with the transition density

$$P'(t,(s,x),\Gamma) = P(s,x,s+t,\Gamma_{s+t}), \tag{4.5}$$

where $\Gamma_{s+t} = \{x \in \mathbb{R}^d : (s+t,x) \in \Gamma\}$.

It is known (see [25]) that $\mathbb{X}'$ admits the strict Fukushima decomposition, i.e., for every $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\mathbf{X}_t = \mathbf{X}_0 + \mathbf{A}_t + \mathbf{M}_t, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.},$$

where $\mathbf{A}$ is a CAF of $\mathbb{X}'$ of zero energy and $\mathbf{M}$ is a MAF of $\mathbb{X}'$ of finite energy. It is also known (see [19]) that

$$p(\mathbf{A}_t)(\omega') = A_{s,s+t}(\omega), \quad p(\mathbf{M}_t)(\omega') = M_{s,s+t}(\omega), \quad t \geq 0 \tag{4.6}$$

for $\omega' = (s,\omega)$, where $p : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection. Set

$$A_t = p(\mathbf{A}_t), \quad M_t = p(\mathbf{M}_t), \quad t \geq 0$$

and for $t \geq 0$ define $\tau(t) : \Omega' \to \mathbb{R}_+$ by putting

$$\tau(t)(\omega') = s + t = \tau(0)(\omega) + t$$

for $\omega' = (s,\omega)$. From now on, we adopt the convention that if ξ is a random variable on Ω, then $\xi(\omega') = \xi(\omega)$ for $\omega' = (s,\omega) \in \Omega'$. With this convention

$$\mathbf{X}_t = (\tau(t), X_{\tau(t)}), \quad t \geq 0.$$

We put

$$\zeta = \inf\{t \geq 0; \mathbf{X}_t \notin \mathbb{R}_+ \times D\}, \quad \zeta_\tau = \zeta \wedge T_\tau, \quad T_\tau = T - \tau(0).$$

Set

$$B_t = \int_0^t \sigma^{-1}(\mathbf{X}_r)\,\mathrm{d}M_r, \quad t \geq 0. \tag{4.7}$$

By (4.1) and (4.6),

$$\langle M^i, M^j \rangle_t = \int_0^t a_{ij}(\mathbf{X}_r)\,\mathrm{d}r, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}$$

for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Therefore, B is a Brownian motion under $P'_{s,x}$ for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. In fact, by [20], it is an $\{\mathcal{F}'_t\}_{t\geq 0}$-Brownian motion.

4.3. Capacity, soft measures, and quasi-continuity

Let $\mathcal{W}$ be the space of all $u \in L^2(\mathbb{R}_+; H^1_0(D))$ such that $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}_+; H^{-1}(D))$ endowed with the usual norm $\|u\|_{\mathcal{W}} = \|u\|_{L^2(\mathbb{R}_+;H^1_0(D))} + \|\frac{\partial u}{\partial t}\|_{L^2(\mathbb{R}_+;H^{-1}(D))}$.

We define the parabolic capacity of an open set $U \subset \mathbb{R}_+ \times D$ as

$$\mathrm{cap}(U) = \inf\{\|u\|_{\mathcal{W}} : u \in \mathcal{W}, u \geq \mathbf{1}_U \text{ a.e. in } \mathbb{R}_+ \times D\}.$$

The parabolic capacity of a Borel set $B \subset \mathbb{R}_+ \times D$ is defined as

$$\mathrm{cap}(B) = \inf\{\mathrm{cap}(U) : U \text{ is an open subset of } \mathbb{R}_+ \times D, B \subset U\}.$$

We also consider the Newtonian capacity on D. For an open set $U \subset D$, we put

$$\mathrm{cap}_N(U) = \inf\{\|u\|_{H^1_0(D)} : u \in H^1_0(D), u \geq \mathbf{1}_U \text{ a.e. in } D\},$$

and for a Borel set $B \subset D$, we put

$$\mathrm{cap}_N(B) = \inf\{\mathrm{cap}_N(U) : U \text{ is an open subset of } D, B \subset U\}.$$

From now on, we say that some property is satisfied for quasi-every (q.e. for short) $x \in D$ (respectively, $(s, x) \in \mathbb{R}_+ \times D$) if it is satisfied except for some Borel subset of D (respectively, $\mathbb{R}_+ \times D$) of cap_N (respectively, cap) capacity zero.

Let μ be a Radon measure on D (respectively, $\mathbb{R}_+ \times D$). Following [6], we say that μ is soft if μ charges no set of cap $_N$ (respectively, cap) capacity zero. In what follows by $\mathcal{M}_0(D)$ (respectively, $\mathcal{M}_0$), we denote the set of all soft measures on D (respectively, $\mathbb{R}_+ \times D$) and by $\mathcal{M}_{0,b}(D)$ (respectively, $\mathcal{M}_{0,b}$) the set of all bounded soft measures on D (respectively, $\mathbb{R}_+ \times D$).

Let B be a Borel subset of $\mathbb{R}_+ \times D$ (respectively, D) and $u : B \to \mathbb{R}$ be a Borel measurable function. We say that u is quasi-continuous if for every $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset B$ such that $\mathrm{cap}(B \setminus F_\varepsilon) < \varepsilon$ (respectively, $\mathrm{cap}_N(B \setminus F_\varepsilon) < \varepsilon$) such that $u_{|F_\varepsilon}$ is continuous. In the paper, we shall mostly work with functions on $B = D$

or $B = D_T$. We adopt the convention that for u defined on $B = D$ (respectively, $B = D_T$) $u(x) = 0$ (respectively, $u(t,x) = 0$) for $x \in \mathbb{R}^d \setminus D$ (respectively, $(t,x) \in (\mathbb{R}_+ \times \mathbb{R}^d) \setminus D_T$). It is well known that u on D_T (respectively, D) is quasi-continuous iff the process $[0, \zeta_\tau) \ni t \to u(\mathbf{X}_t)$ (respectively, $[0, \zeta^0) \ni t \to u(X_t)$) is continuous under $P'_{s,x}$ (respectively, $P_{0,x}$) for q.e. $(s,x) \in D_T$ (respectively, q.e. $x \in D$). We will also consider quasi-càdlàg functions on D_T, i.e., Borel functions u on D_T, such that the process $[0, \zeta_\tau] \ni t \to u(\mathbf{X}_t)$ is càdlàg for q.e. $(s,x) \in D_T$.

4.4. Additive functionals and soft measures

Let $E_{s,x}$ (respectively, $E'_{s,x}$) denote the expectation with respect to $P_{s,x}$ (respectively, $P'_{s,x}$) and let m_1 denote the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Let ζ^0 be defined by (1.7) with $s = 0$, i.e.,

$$\zeta^0 = \inf\{t \geq 0,\ X_t \notin D\}.$$

Let us recall that a positive AF A of $\mathbb{X}'$ and a positive soft measure μ on $\mathbb{R}_+ \times D$ are in the Revuz correspondence if

$$\langle \mu, f \rangle \equiv \int_0^\infty \int_D f \, \mathrm{d}\mu = \lim_{\alpha \to \infty} \alpha \int_0^\infty \int_D \left(E'_{s,x} \int_0^\zeta \mathrm{e}^{-\alpha t} f(\mathbf{X}_t) \, \mathrm{d}A_t \right) \mathrm{d}m_1(s,x) \tag{4.8}$$

for every $f \in \mathcal{B}^+(\mathbb{R}_+ \times D)$. If $\langle \mu, 1 \rangle < \infty$, then A is called integrable. It is known (see [22,30]) that under (4.8), the family of all integrable positive AFs of $\mathbb{X}'$ and the family of all bounded positive soft measures on $\mathbb{R}_+ \times D$ are in one-to-one correspondence. In what follows the additive functional corresponding to a positive bounded soft measure μ will be denoted by A^μ.

Let p'_D denote the transition density of the process $\mathbb{X}'$ killed on exiting $\mathbb{R}_+ \times D$. It is known that A^μ corresponds to μ iff for q.e. $(s,x) \in \mathbb{R}_+ \times D$,

$$E'_{s,x} \int_0^\zeta f(\mathbf{X}_t) \, \mathrm{d}A^\mu_t = \int_0^\infty \left(\int_0^\infty \int_D f(z) p'_D(t, (s,x), z) \, \mathrm{d}\mu(z) \right) \mathrm{d}t \tag{4.9}$$

for every $f \in \mathcal{B}_+(\mathbb{R}^+ \times D)$ (see [22]).

Suppose now that the coefficients of the operator (1.2) do not depend on time, i.e., $a_{ij}(t,x) = a_{ij}(x)$ for $(t,x) \in D_T$. We then set

$$A = \sum_{ij=1}^d \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial}{\partial x_i} \right). \tag{4.10}$$

It is well known that the fundamental solution for A has the property that $p(s,x,t,y) = p(t-s,x,y)$ for any $t > s$, $x \neq y$. Consequently, $\mathbb{X} = \{(X, P_x) : x \in \mathbb{R}^d\}$, where $P_x = P'_{0,x}$ is a time-homogeneous Markov process with the transition density $p(t,x,y) = p(0,x,t,y)$. It is also known (see, e.g., [9]) that in the time-homogeneous case, a positive CAF A of $\mathbb{X}$ and a positive soft measure μ on D are in the Revuz correspondence if

$$\langle \mu, f\rangle = \lim_{\alpha\to\infty} \alpha \int_D \left(E_x \int_0^{\zeta^0} \mathrm{e}^{-\alpha t} f(X_t)\,\mathrm{d}A_t \right) \mathrm{d}m(x) \tag{4.11}$$

for every $f \in \mathcal{B}^+(D)$, where E_x denotes the expectation with respect to P_x. If $\langle \mu, 1\rangle < \infty$, then A is called integrable. In [9], it is proved that the family of all integrable positive CAFs of $\mathbb{X}$ and the family of all bounded positive soft measures on D are in one-to-one correspondence via formula (4.11).

Let A^μ denote the additive functional corresponding to a positive bounded soft measure μ, p_D denote the transition density of the process $\mathbb{X}$ killed on exiting D, and let $G_D(x, y) = \int_0^\infty p_D(t, x, y)\,\mathrm{d}t$ be the Green function for D. Then, for q.e. $x \in D$,

$$E_x \int_0^{\zeta^0} f(X_t)\,\mathrm{d}A_t^\mu = \int_0^\infty \int_D f(y) p_D(t, x, y)\,\mathrm{d}\mu(y)\,\mathrm{d}t = \int_D f(y) G_D(x, y)\,\mathrm{d}\mu(y) \tag{4.12}$$

for every $f \in \mathcal{B}^+(D)$. In the whole paper for a fixed Borel positive measure μ on D_T (respectively, D), we denote

$$R^{0,T}\mu(s, x) = \int_0^\infty \left(\int_0^T \int_D p'_D(t, (s, x), z)\,\mathrm{d}\mu(z) \right) \mathrm{d}t, \quad R\mu(x) = \int_D G_D(x, y)\,\mathrm{d}\mu(y).$$

Finally, let us recall that there is c depending only on d, Λ such that $p(t, x, y) \le Ct^{-d/2}$ for $t > 0$, $x, y \in \mathbb{R}^d$ (see, e.g., [1]). Therefore, by [5, Theorem 1.17], there is c depending only on d, Λ such that

$$\sup_{x\in\mathbb{R}^d} E_x \zeta^0 \le c|D|^{d/2}, \tag{4.13}$$

where $|D|$ denotes the Lebesgue measure of D, whereas from Corollary to Proposition 1.18 in [5], it follows that there exists constants $a > 0$, $b > 0$ depending only on d, Λ and $|D|$ such that for every $t > 0$,

$$\sup_{x\in\mathbb{R}^d} P_x(\zeta^0 > t) \le a\mathrm{e}^{-bt}. \tag{4.14}$$

5. Markov-type BSDEs and PDEs

Let us fix $T > 0$ and set $D_T = [0, T] \times D$. In this section, we show existence and uniqueness results for systems of PDEs of the form

$$\begin{cases} \frac{\partial u^k}{\partial t} + A_t u^k = -f^k(t, x, u) - \mu^k, \quad k = 1, \dots, N, \\ u_{|\partial D}(t, \cdot) = 0, \ t \in [0, T), \quad u(T, \cdot) = \varphi \quad \text{on } D, \end{cases} \tag{5.1}$$

where A_t is given by (1.2).

Let $f : D_T \times \mathbb{R}^N \to \mathbb{R}^N$. We consider the following hypotheses.

(H1) $f(\cdot,\cdot,y)$ is measurable for every $y\in\mathbb{R}^N$ and $f(t,x,\cdot)$ is continuous for a.e. $(t,x)\in D_T$.
(H2) There is $\alpha\in\mathbb{R}$ such that $\langle f(t,x,y)-f(t,x,y'),y-y'\rangle\le\alpha|y-y'|^2$ for every $y,y'\in\mathbb{R}^N$ and a.e. $(t,x)\in D_T$.
(H3) $f(\cdot,\cdot,0)\in L^1(D_T)$, $\mu\in\mathcal{M}_{0,b}(D_T)$, $\varphi\in L^1(D)$.
(H4) $\forall_{r>0,\,y\in\mathbb{R}^N}\;\; R^{0,T}(\sup_{|y|\le r}|f(\cdot,\cdot,y)|)<\infty$, m_1-a.e.

REMARK 5.1. It is known (see [17, Proposition 3.6]) that if $f\in L^1(D_T)$, then $R^{0,T}|f|<\infty$, m_1-a.e. Therefore, (H4) is satisfied if $\sup_{|y|\le r}|f(\cdot,\cdot,y)|\in L^1(D_T)$ for every $y\in\mathbb{R}^N$ and $r\ge 0$. However, the class of functions $f\in\mathcal{B}(D_T)$ such that $R^{0,T}|f|<\infty$, m_1-a.e. is wider than $L^1(D_T)$. It includes in particular the space $L^1(D_T;\delta\cdot m_1)$, where $\delta(x)=\mathrm{dist}(x,\partial D)$ (see [17, Example 5.2]).

It is known that in case $N=1$, one can find solutions of problems of the form (5.1) in the (nonlinear) space $\mathcal{T}_2^{0,1}$ of all Borel measurable functions u on D_T such that $T_k(u)\in L^2(0,T;H_0^1(D))$ for every $k\ge 1$ (see [2,6]). In the case of systems, the problem is more difficult, because we do not know whether the solutions or its truncation have gradients in the usual sense (i.e., locally in some Sobolev space). This is related to the lack of integrability of f_u. The same problem appears in the case of elliptic systems. In the scalar case, it is known that a solution of (5.1) belongs to $W_0^{1,q}(D)$ for every $q\in[1,\frac{d}{d-1})$ (see [34]) but in case $N>1$, the problem whether a solution belongs to $W_0^{1,q}(D)$ for some $q\ge 1$ is open. To overcome the difficulty in [15], the existence and uniqueness of elliptic systems with measure data similar to (5.1) is proved in some wider than $\mathcal{T}_2^{0,1}$ (see Corollary 5.6) linear space. The space introduced in [15] makes essential use of the Markov process $\mathbb{X}$ associated with the operator A defined by (4.10) and therefore may be called a stochastic Sobolev space. In what follows we extend the ideas from [15] to parabolic systems. We begin with the definition of the stochastic Sobolev space of functions depending on time.

Let $W^{0,1}(\mathbb{X}^{D_T})$ denote the set of all $u\in FM$ (definition below) for which there exists a sequence $\{u_n\}\subset C_c^\infty(D_T)$ such that for q.e. $(s,x)\in D_T$,

$$\int_0^{\zeta_\tau}|(u_n-u)(\mathbf{X}_t)|^2\,\mathrm{d}t\to 0\quad\text{in probability } P'_{s,x}\text{ as } n\to\infty \tag{5.2}$$

and

$$\int_0^{\zeta_\tau}|\nabla(u_n-u_m)(\mathbf{X}_t)|^2\,\mathrm{d}t\to 0\quad\text{in probability } P'_{s,x}\text{ as } n,m\to\infty. \tag{5.3}$$

In [14], it is proved that for every $u\in W^{0,1}(\mathbb{X}^{D_T})$, there exists a unique (a.e.) function $v\in\mathcal{B}(D_T)$ such that for every $\{u_n\}\subset C_c^\infty(D_T)$ satisfying (5.2) and (5.3),

$$\int_0^{\zeta_\tau}|\nabla u_n(\mathbf{X}_t)-v(\mathbf{X}_t)|^2\,\mathrm{d}t\to 0\quad\text{in probability } P'_{s,x}\text{ as } n\to\infty \tag{5.4}$$

for q.e. $(s,x)\in D_T$.

Given $u \in W^{0,1}(\mathbb{X}^{D_T})$, we denote by $\nabla_{\mathbb{X}} u$ the unique function v satisfying (5.4). Notice that directly from the construction of $\nabla_{\mathbb{X}} u$, it follows that $\nabla_{\mathbb{X}} u = \nabla u$ a.e. if $u \in L^2(0, T; H_0^1(D))$.

By FM, we denote the space of Borel measurable functions u on D_T such that for q.e. $(s, x) \in D_T$, $P'_{s,x}(\int_0^{\zeta_\tau} |u(\mathbf{X}_r)|^2 \,\mathrm{d}r < \infty) = 1$. We say that $u_n \to u$ in FM if (5.2) holds for q.e. $(s, x) \in D_T$.

DEFINITION. Let $u_n, u \in W^{0,1}(\mathbb{X}^{D_T})$. We say that $u_n \to u$ in $W^{0,1}(\mathbb{X}^{D_T})$ if $u_n \to u$ in FM and $\nabla_{\mathbb{X}} u_n \to \nabla_{\mathbb{X}} u$ in FM.

REMARK 5.2. (i) Let $\mathcal{O}(\rightarrow)$ be the topology generated by the convergence in $W^{0,1}(\mathbb{X}^{D_T})$. This topology is metrizable by the F-norm

$$|u|_1 = E_{m_1}\left[\left(\int_0^{\zeta_\tau} u^2(\mathbf{X}_r)\,\mathrm{d}r\right)^{1/2} \wedge 1\right] + E_{m_1}\left[\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u|^2(\mathbf{X}_r)\,\mathrm{d}r\right)^{1/2} \wedge 1\right].$$

Indeed, by the Kantorovich–Kisyński theorem (see [11,12]),

$$u_n \to u \text{ in } \mathcal{O}(\rightarrow) \text{ iff } \forall (n_k) \subset (n)\ \exists (n_{k_l}) \subset (n_k)\quad u_{n_{k_l}} \to u \text{ in } W^{0,1}(\mathbb{X}^{D_T}). \tag{5.5}$$

Assume that $u_n \to u$ in $\mathcal{O}(\rightarrow)$ and (u_n) does not converge to u in $|\cdot|_1$. Then, there exists $\varepsilon > 0$ and subsequence $(n_k) \subset (n)$ such that

$$|u - u_{n_k}|_1 > \varepsilon, \quad k \geq 1. \tag{5.6}$$

On the other hand, by (5.5), there exists a subsequence $(n_{k_l}) \subset (n_k)$ such that $u_{n_{k_l}} \to u$ in $W^{0,1}(\mathbb{X}^{D_T})$. Hence, by the Lebesgue dominated convergence theorem, $|u_{n_{k_l}} - u|_1 \to 0$ as $l \to \infty$, which contradicts (5.6). Now, assume that $|u_n - u|_1 \to 0$ as $n \to \infty$ and let $(n_k) \subset (n)$. By [16, Proposition 3.3], there exists a subsequence $(n_{k_l}) \subset (n_k)$ such that $u_{n_{k_l}} \to u$ in $W^{0,1}(\mathbb{X}^{D_T})$. Since $(n_k) \subset (n)$ was arbitrary, (5.5) implies that $u_n \to u$ in $\mathcal{O}(\rightarrow)$.

(ii) By [14, Proposition 4.6], the space $(W^{0,1}(\mathbb{X}^{D_T}), \mathcal{O}(\rightarrow))$ is complete.

LEMMA 5.3. *If $u \in W^{0,1}(\mathbb{X}^{D_T})$, $\theta \in C^1(\mathbb{R})$ and there is $c > 0$ such that $|\theta'(t)| \leq c$ for $t \in \mathbb{R}$, then $\theta(u) \in W^{0,1}(\mathbb{X}^{D_T})$ and $\nabla_{\mathbb{X}}(\theta(u)) = \theta'(u)\nabla_{\mathbb{X}} u$.*

Proof. Since $u \in W^{0,1}(\mathbb{X}^{D_T})$, there exists a sequence $\{\eta_n\} \subset C_c(D_T)$ such that $\eta_n \to u$ and $\nabla \eta_n \to \nabla_{\mathbb{X}} u$ in FM. Using the assumptions on θ, one can easily show that $\theta(\eta_n) \to \theta(u)$ and $\nabla\theta(\eta_n) = \theta'(\eta_n)\nabla\eta_n \to \theta'(u)\nabla_{\mathbb{X}} u$ in FM, which proves the desired result. □

LEMMA 5.4. *Let $k \in \mathbb{R}$ and $u \in W^{0,1}(\mathbb{X}^{D_T})$. Then, $u \wedge k, u \vee k \in W^{0,1}(\mathbb{X}^{D_T})$ and*

$$\nabla_{\mathbb{X}}(u \wedge k) = \mathbf{1}_{(-\infty,k)}(u)\nabla_{\mathbb{X}} u = \mathbf{1}_{(-\infty,k]}(u)\nabla_{\mathbb{X}} u, \quad \text{a.e.},$$
$$\nabla_{\mathbb{X}}(u \vee k) = \mathbf{1}_{(k,\infty)}(u)\nabla_{\mathbb{X}} u = \mathbf{1}_{[k,\infty)}(u)\nabla_{\mathbb{X}} u, \quad \text{a.e.}$$

Proof. We will prove the lemma for $u \wedge k$. The proof for $u \vee k$ is analogous. Set

$$\sigma_n(t) = \begin{cases} 1, & \text{if } t \le k, \\ n(k-t)+1, & \text{if } t \in \left(k, k+\frac{1}{n}\right), \\ 0, & \text{if } t \ge k+\frac{1}{n} \end{cases}$$

and $\theta_n(t) = \int_0^t \sigma_n(r)\,\mathrm{d}r,\ t \in \mathbb{R}$. Standard arguments show that $\theta_n(u) \to u \wedge k$ in FM and, by the use of Lemma 5.3, that $\nabla_{\mathbb{X}}\theta_n(u) = \sigma_n(u)\nabla_{\mathbb{X}}u \to \mathbf{1}_{(-\infty,k)}(u)\nabla_{\mathbb{X}}u$ in FM. Hence, $u \wedge k \in W^{0,1}(\mathbb{X}^{D_T})$ and $\nabla_{\mathbb{X}}(u \wedge k) = \mathbf{1}_{(-\infty,k)}(u)\nabla_{\mathbb{X}}u$. If we repeat the above arguments with σ_n replaced by $\hat{\sigma}_n$ defined as

$$\hat{\sigma}_n(t) = \begin{cases} 1 & \text{if } t \le k-\frac{1}{n} \\ n\left(k-\frac{1}{n}-t\right)+1 & \text{if } t \in \left(k-\frac{1}{n}, k\right), \\ 0 & \text{if } t \ge k \end{cases}$$

we will obtain $\nabla_{\mathbb{X}}(u \wedge k) = \mathbf{1}_{(-\infty,k]}(u)\nabla_{\mathbb{X}}u$. □

Let FB denote the space of Borel measurable functions on D_T such that for q.e. $(s,x) \in D_T,\ P'_{s,x}(\operatorname{ess\,sup}_{r\in[0,\zeta_\tau]}|u(\mathbf{X}_r)| < \infty) = 1$. Observe that every quasi-càdlàg function belongs to FB.

One of the reason why the space $\mathcal{T}_2^{0,1}$ has been introduced is that the standard Sobolev space $L^2(0,T;H_0^1(D))$ lacks the property that $u \in L^2(0,T;H_0^1(D))$ if u is quasi-càdlàg (natural class for solutions of equations with measure data) and $T_k(u) \in L^2(0,T;H_0^1(D))$ for every $k \ge 0$. The following lemma shows the stochastic Sobolev space has this remarkable feature.

LEMMA 5.5. *If $u \in FB$ and $T_k(u) \in W^{0,1}(\mathbb{X}^{D_T})$ for every $k \ge 0$, then $u \in W^{0,1}(\mathbb{X}^{D_T})$.*

Proof. First, observe that for q.e. $(s,x) \in D_T$ and $\varepsilon > 0$,

$$P'_{s,x}\left(\int_0^{\zeta_\tau} |T_k(u)-u|^2(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \le P'_{s,x}\left(2\int_0^{\zeta_\tau} |u|^2\mathbf{1}_{\{|u|>k\}}(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \to 0,$$

which shows that $T_k(u) \to u$ in FM. By Lemma 5.4, for $k < l$, we have

$$\begin{aligned} &P'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}}T_k(u)-\nabla_{\mathbb{X}}T_l(u)|^2(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \\ &\quad = P'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}}T_k(T_l(u))-\nabla_{\mathbb{X}}T_l(u)|^2(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \\ &\quad = P'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}}T_l(u)\mathbf{1}_{\{|T_l(u)|<k\}}-\nabla_{\mathbb{X}}T_l(u)|^2(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \\ &\quad = P'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}}T_l(u)|^2\mathbf{1}_{\{|u|\ge k\}}(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right). \end{aligned}$$

Observe that

$$P'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}}T_l(u)|^2\mathbf{1}_{\{|u|\ge k\}}(\mathbf{X}_r)\,\mathrm{d}r > \varepsilon\right) \le P'_{s,x}(\operatorname*{ess\,sup}_{r\in[0,\zeta_\tau]}|u(\mathbf{X}_r)| \ge k).$$

By the assumption that $u \in FB$, the right-hand side of the above inequality tends to zero as $k \to \infty$, which shows that $\nabla_{\mathbb{X}} T_k(u) - \nabla_{\mathbb{X}} T_l(u) \to 0$ in FM as $k, l \to \infty$. Consequently, $u \in W^{0,1}(\mathbb{X}^{D_T})$. $\square$

In [14], it is shown by examples that in general neither $\mathcal{T}_2^{0,1} \subset W^{0,1}(\mathbb{X}^{D_T})$ nor $\mathcal{T}_2^{0,1} \supset W^{0,1}(\mathbb{X}^{D_T})$. However, we have the following corollary to Lemma 5.5.

COROLLARY 5.6. *If $u \in \mathcal{T}_2^{0,1}$ and $u \in FB$, then $u \in W^{0,1}(\mathbb{X}^{D_T})$.*

Besides being nonlinear, another drawback to the space $\mathcal{T}_2^{0,1}$ is that it is sometimes too small in practice. For instance, in [14], it is proved that solutions of some types of the obstacle problem are quasi-continuous and belong to $W^{0,1}(\mathbb{X}^{D_T})$ but do not belong to $\mathcal{T}_2^{0,1}$.

EXAMPLE 5.7. Let $B(0, k^{-1}) = \{x \in \mathbb{R}^2 : |x| < k^{-1}\}$, $D = B(0, 1)$, and let $\alpha \in \mathbb{R}$.

(i) Put $v(t, x) = 1 - |x|^{-\alpha}$ for $(t, x) \in D_T$. Since $\mathrm{cap}_N(B(0, k^{-1})) \to 0$ as $k \to \infty$, $v \in W^{0,1}(\mathbb{X}^{D_T})$ and

$$\nabla_{\mathbb{X}} v(t, x) = -\alpha x |x|^{-(\alpha+2)}, \quad (t, x) \in D_T$$

for every $\alpha \in \mathbb{R}$. If $\alpha \geq 2$, then v is not locally integrable, so there is no sense to speak about its distributional derivative. But let us observe that $v \in \mathcal{T}_2^{0,1}$ for every $\alpha \in \mathbb{R}$.

(ii) Put $u(t, x) = \sin v(t, x)$, $(t, x) \in D_T$. Then, by Lemma 5.3, $u \in W^{0,1}(\mathbb{X}^{D_T})$ and

$$\nabla_{\mathbb{X}} u(t, x) = \alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha})$$

for every $\alpha \in \mathbb{R}$. One can check that for every $\varepsilon > 0$ and $\alpha > 1$,

$$\int_{B(0,\varepsilon)} \alpha |x|^{-(\alpha+1)} |\cos(1 - |x|^{-\alpha})| \, \mathrm{d}x = \infty,$$

which shows that if $\alpha > 1$, then $u \notin \mathcal{T}_2^{0,1}$ since $T_k(u) = u$ for $k \geq 1$. However, $u \in L^1(D_T)$ and one can check that its distributional derivative is given by the formula

$$\nabla u = \mathrm{p.v.} \left(\alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha}) \right),$$

i.e., for every $\eta \in C_0^\infty(D)$,

$$(\nabla u)(\eta) = \lim_{k \to \infty} \int_{D \setminus B(0,k^{-1})} \alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha}) \eta(x) \, \mathrm{d}m(x).$$

Accordingly, even if the distributional derivative of $u \in W^{0,1}(\mathbb{X}^{D_T})$ exists, it is not a function in general.

Following [18], we adopt the following definition.

DEFINITION. We say that a measurable function $f : D_T \to \mathbb{R}$ is quasi-integrable if the function $[0, \zeta_\tau] \ni t \mapsto f(\mathbf{X}_t)$ belongs to $L^1([0, \zeta_\tau])$ $P'_{s,x}$-a.s. for q.e. $(s, x) \in D_T$. The set of all quasi-integrable functions on D_T will be denoted by $qL^1(D_T)$.

We say that a measurable function u on D_T is of class (FD) if for q.e. $(s, x) \in D_T$, the process $u(\mathbf{X})$ on $[0, \zeta_\tau]$ is of class (D) under the measure $P'_{s,x}$.

REMARK 5.8. It is known (see [19]) that if f is quasi-integrable in the analytic sense, i.e., if for every $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D_T$ such that $\mathrm{cap}(G_\varepsilon) < \varepsilon$ and $f_{|D_T\setminus G_\varepsilon} \in L^1(D_T\setminus G_\varepsilon)$ then $f \in qL^1$. In particular, it follows that if $f \in L^1(D_T)$, then $f \in qL^1(D_T)$. It is also clear that if $R^{0,T}|f| < \infty$, m_1-a.e., then $f \in qL^1(D_T)$.

DEFINITION. We say that a measurable function $u : D_T \to \mathbb{R}^N$ is a solution of system (5.1) if

(a) $(t, x) \mapsto f(t, x, u(t, x)) \in qL^1(D_T)$, $u \in W^{0,1}(\mathbb{X}^{D_T})$ and u is of class (FD),
(b) For q.e. $(s, x) \in D_T$,

$$
\begin{aligned}
u(\mathbf{X}_t) = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, u(\mathbf{X}_r))\, \mathrm{d}r \\
+ \int_t^{\zeta_\tau} \mathrm{d}A_r^\mu - \int_t^{\zeta_\tau} \sigma\nabla_{\mathbb{X}} u(\mathbf{X}_r)\, \mathrm{d}B_r, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}
\end{aligned}
\tag{5.7}
$$

In what follows for a given function u on D_T, we set

$$
\bar{u}(t, x) = u(T - t, x), \quad (t, x) \in D_T,
$$

and given $\mu \in \mathcal{M}_{0,b}(D_T)$, we denote by $\bar{\mu}$ the unique measure in $\mathcal{M}_{0,b}(D_T)$ such that

$$
\int_{D_T} \bar{\eta}\, \mathrm{d}\mu = \int_{D_T} \eta\, \mathrm{d}\bar{\mu}, \quad \eta \in \mathcal{B}_b(D_T).
$$

DEFINITION. We say that u is a solution of (1.1) on $[0, T]$ if $\bar{u}$ is a solution of (5.1) with f_u replaced by $\bar{f}_u$, μ replaced by $\bar{\mu}$ and a replaced by $\bar{a}$.

DEFINITION. Let $(s, x) \in D_T$. We say that a pair $(Y^{s,x}, Z^{s,x})$ consisting of an $\mathbb{R}^N$-valued process $Y^{s,x}$ and an $\mathbb{R}^d \times \mathbb{R}^N$-valued process $Z^{s,x}$ is a solution of $\mathrm{BSDE}_{s,x}(\varphi, D, f + \mathrm{d}\mu)$ if $Y^{s,x}, Z^{s,x}$ are $\{\mathcal{F}'_t\}$ progressively measurable, $Y^{s,x}$ is càdlàg, $t \mapsto f(\mathbf{X}_t, Y_t^{s,x}, Z_t^{s,x}) \in L^1(0, \zeta_\tau)$, $P'_{s,x}$-a.s., $P'_{s,x}(\int_0^{\zeta_\tau} |Z_r^{s,x}|^2\, \mathrm{d}r < \infty) = 1$ and

$$
\begin{aligned}
Y_t^{s,x} = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, Y_r^{s,x}, Z_r^{s,x})\, \mathrm{d}r \\
+ \int_t^{\zeta_\tau} \mathrm{d}A_r^\mu - \int_t^{\zeta_\tau} Z_r^{s,x}\, \mathrm{d}B_r, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}
\end{aligned}
$$

Observe that from the above two definitions, it follows that if u is a solution of (5.1), then for q.e. $(s,x)\in D_T$ the pair $(Y^{s,x},Z^{s,x})=(Y,Z)$, where

$$(Y_t,Z_t)=(u(\mathbf{X}_t),\sigma\nabla_{\mathbb{X}}u(\mathbf{X}_t)),\quad t\in[0,\zeta_\tau]$$

is a solution of $\mathrm{BSDE}_{s,x}(\varphi,D,f+\mathrm{d}\mu)$. In the rest of this section, we are going to prove that under (H1)–(H4), this statement can be reversed in the following sense. For q.e. $(s,x)\in D_T$, there exists a unique solution $(Y^{s,x},Z^{s,x})$ of $\mathrm{BSDE}_{s,x}(\varphi,D,f+\mathrm{d}\mu)$ and if we set $u(s,x)=E'_{s,x}Y_0^{s,x}$ for $(s,x)\in D_T$, then u is a solution of (5.1). Moreover,

$$Y_t^{s,x}=u(\mathbf{X}_t),\ t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s.},\quad Z^{s,x}=\sigma\nabla_{\mathbb{X}}u(\mathbf{X}),\ \mathrm{d}t\otimes P'_{s,x}\text{-a.s.}\tag{5.8}$$

Let us first state the following corollary to Theorem 3.7.

PROPOSITION 5.9. *Assume* (H1)–(H4). *Let F be the set of those $(s,x)\in D_T$ for which the data ζ_τ, $f(\mathbf{X},\cdot,\cdot)$, $\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau\}}$, A^μ satisfy assumptions* (A1)–(A4) *under the measure $P'_{s,x}$. Then, $\mathrm{cap}(D_T\setminus F)=0$, and for every $(s,x)\in F$, there exists a unique solution $(Y^{s,x},Z^{s,x})$ of* $\mathrm{BSDE}_{s,x}(\varphi,D,f+\mathrm{d}\mu)$ *such that $(Y^{s,x},Z^{s,x})\in\mathcal{D}^q\otimes M^q$ for $q\in(0,1)$ and $Y^{s,x}$ is of class* (D).

Proof. That $\mathrm{cap}(D_T\setminus F)=0$ follows from [16, Corollary 3.4] (see also [13, Remark 3.2]). The second assertion follows from Theorem 3.7. □

The crucial issue in the proof of representation (5.8) and existence of solution to system (5.1) is regularity of the function $D_T\ni(s,x)\mapsto u(s,x)=E'_{s,x}Y_0^{s,x}$. In most papers concerning probabilistic solutions for PDEs or stochastic representation for solutions of PDEs, the regularity is proved by using the results of probabilistic potential theory which may be applied when $u(s,x)=E'_{s,x}\int_0^{\zeta_\tau}\mathrm{d}A_r$ for some AF of $\mathbb{X}'$. Here, such approach cannot be applied because in general, u does not admit the last representation [in general, integrals on the right-hand side of (1.8) do not exists]. We cope with the problem of regularity of u in Propositions 5.10, 5.11.

PROPOSITION 5.10. *Let F be a Borel subset of D_T such that* $\mathrm{cap}(D_T\setminus F)=0$. *Assume that for every $(s,x)\in F$, the real process $Y^{s,x}$ is a continuous semimartingale under $P'_{s,x}$ such that $Y^{s,x}_{t\vee\zeta}=0$, $t\geq 0$, and there exists a Borel function v on D_T such that for every $(s,x)\in F$ and every $t\in[0,T_\tau]$,*

$$v(\mathbf{X}_t)=Y_t^{s,x},\quad P'_{s,x}\text{-a.s.}\tag{5.9}$$

Then, $u(s,x)=E'_{s,x}Y_0^{s,x}$ is a quasi-continuous version of v and for every $(s,x)\in F$,

$$u(\mathbf{X}_t)=Y_t^{s,x},\quad t\in[0,\zeta_\tau],\quad P'_{s,x}\text{-a.s.}$$

Proof. Let $(s,x)\in F$. Since $Y^{s,x}$ is a continuous semimartingale, there exists a finite variation continuous process $R^{s,x}$ and $Z^{s,x}\in M$ such that

$$Y_t^{s,x}=\mathbf{1}_{\{\zeta>T_\tau\}}v(\mathbf{X}_{T_\tau})+\int_t^{\zeta_\tau}\mathrm{d}R_r^{s,x}-\int_t^{\zeta_\tau}Z_r^{s,x}\,\mathrm{d}B_r,\quad t\in[0,\zeta_\tau],\quad P'_{s,x}\text{-a.s.}$$

First, let us assume additionally that $Y^{s,x}$ is bounded. Write $L^n = Y^{s,x} - \frac{1}{n}$ and $U = Y^{s,x}$. Then, $L^n < U$ and by [10], for every $(s, x) \in F$, there exists a unique solution (Y^n, Z^n, R^n), on the space $(\Omega', \mathcal{F}'_\infty, P'_{s,x})$, of the reflected BSDE with final condition $\xi \equiv v(\mathbf{X}^D_{T_\tau})$, lower barrier L^n and upper barrier U (RBSDE$_{s,x}(\xi, 0, L^n, U)$ for short) such that $Y^n \in \mathcal{S}^2$, $Z^n \in M$ and $R^n \in \mathcal{V}$. By [10, Theorem 1.3], $Y^n_t \le Y^{n+1}_t$, $t \in [0, \zeta_\tau]$, $P'_{s,x}$-a.s. for every $n \ge 1$. Therefore,

$$Y^n_t \nearrow Y_t, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \tag{5.10}$$

Let us fix $n \in \mathbb{N}$ and set $H_t = Y^n_t$, $t \in [0, \zeta_\tau]$. By [10],

$$H^k_t \nearrow H_t, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.,} \tag{5.11}$$

where H^k is the first component of the solution of RBSDE$_{s,x}(0, f_k, U)$ with $f_k(t, y) = k(y - L^n_t)^-$. Let $H^{k,l}$ denote the first component of the solution of BSDE$_{s,x}(v(T, \cdot), f_{k,l})$ with $f_{k,l}(t, y) = k(y - L^n_t)^- - l(y - U_t)^+$. Then, by [7],

$$H^{k,l}_t \searrow H^k_t, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \tag{5.12}$$

By (5.9), $f_{k,l}(\cdot, y) = k(y - v(\mathbf{X}) + \frac{1}{n})^- - l(y - v(\mathbf{X}))^+$, $\mathrm{d}t \otimes P'_{s,x}$-a.e. on $[0, \zeta_\tau] \times \Omega$ for every $y \in \mathbb{R}$. Set $g_{k,l}(t, x, y) = k(y - v(x) + \frac{1}{n})^- - l(y - v(x))^+$. By [13],

$$H^{k,l}_t = h_{k,l}(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.,}$$

where $h_{k,l}$ is a quasi-continuous version of the solution of PDE$(0, g_{k,l})$. Let us put

$$h(s, x) = \limsup_{k \to \infty} \liminf_{l \to \infty} h_{k,l}(s, x), \quad (s, x) \in D_T.$$

Then, by (5.11) and (5.12),

$$H_t = h(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

for every $(s, x) \in F$. In particular, since $H \in \mathcal{S}^2$, the function h is quasi-continuous. From what has already been proved it follows that for every $n \ge 1$, there exists a quasi-continuous function u_n such that

$$Y^n_t = u_n(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

for every $(s, x) \in F$. Putting $u(s, x) = \limsup_{n \to \infty} u_n(s, x)$, $(s, x) \in D_T$, we get by (5.10) the desired result.

Now, we show how to dispense with the assumption that $Y^{s,x}$ is bounded. By the Itô–Tanaka formula, $T_k(Y)$ is a semimartingale, so by the first part of the proof, the function u_k defined as $u_k(s, x) = E'_{s,x} T_k(Y^{s,x}_0)$, $(s, x) \in D_T$, is quasi-continuous and for each $k \in \mathbb{N}$,

$$T_k(Y^{s,x}_t) = u_k(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \tag{5.13}$$

for every $(s, x) \in F$. But $u_k(s, x) = E_{s,x} T_k(Y^{s,x}_0) = T_k(E_{s,x} Y^{s,x}_0)$ since $Y^{s,x}_0$ is constant $P'_{s,x}$-a.s. Therefore, letting $k \to \infty$ in (5.13), we get the desired result for Y. □

PROPOSITION 5.11. *Let F be a Borel subset of D_T such that* $\mathrm{cap}(D_T \setminus F) = 0$ *and let $u : D_T \to \mathbb{R}$ be a Borel function. If $u(\mathbf{X})$ is a continuous semimartingale under $P'_{s,x}$ on $[0, \zeta_\tau]$ for $(s, x) \in F$, then $u \in W^{0,1}(\mathbb{X}^{D_T})$ and there exists CAF A of finite variation such that*

$$u(\mathbf{X}_t) = u(s, x) + \int_0^t \mathrm{d}A_r + \int_0^t \sigma \nabla_{\mathbb{X}} u(\mathbf{X}_r)\,\mathrm{d}B_r, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

for every $(s, x) \in F$.

Proof. We adopt the notation of the proof of Proposition 5.10 and put $M^n_t = \int_0^t Z^n_r\,\mathrm{d}B_r$. Let us first assume that u is bounded. By [14], we know that

$$Y^n_t = u_n(\mathbf{X}_t),\ t \in [0, \zeta_\tau], \quad \sigma \nabla_{\mathbb{X}} u_n(\mathbf{X}) = Z^n,\ \mathrm{d}t \otimes P'_{s,x}\text{-a.e.}, \tag{5.14}$$

where $u_n \in W^{0,1}(\mathbb{X}^{D_T})$ is a solution of OP $(v(T, \cdot), 0, L^n, U)$ (with two barriers). Fix $(s, x) \in F$. Since $Y = u(\mathbf{X})$ is a continuous semimartingale and the underlying filtration is Brownian, there exists a finite variation continuous process $R^{s,x}$ and a process $Z^{s,x} \in M$ such that

$$Y_t = Y_0 + \int_0^t \mathrm{d}R^{s,x}_r + \int_0^t Z^{s,x}_r\,\mathrm{d}B_r, \quad t \in [0, \zeta_\tau]. \tag{5.15}$$

Put $M^{s,x}_t = \int_0^t Z^{s,x}_r\,\mathrm{d}B_r$. Since the process Y is continuous, from the proof of Proposition 5.10 and Dini's theorem, it follows that

$$P'_{s,x}\big(\sup_{t \in [0,\zeta_\tau]} |Y^n_t - Y_t|^2 > \varepsilon\big) \to 0. \tag{5.16}$$

Moreover, by [14, Proposition 6.1],

$$\mathrm{d}R^{n,+} \le \mathbf{1}_{\{Y^n_t = L^n_t\}} \mathrm{d}R^+_t, \quad \mathrm{d}R^{n,-} \le \mathbf{1}_{\{Y^n_t = U_t\}} \mathrm{d}R^-_t.$$

Therefore, there exists a sequence of stopping times $\{\tau_k\}$ such that $\tau_k \le \tau_{k+1}$, $k \ge 1$, $\tau_k \to \zeta_\tau$ $P'_{s,x}$-a.s. for q.e. $(s, x) \in D_T$ and for every $k \ge 1$, the sequence $\{Y^{n,\tau_k}\}$ satisfies the condition UT (see, e.g., [33]). Therefore,

$$P'_{s,x}(\langle M^n - M^{s,x}\rangle_{\zeta_\tau} > \varepsilon) \to 0 \tag{5.17}$$

(see, e.g., [33, Proposition 1.5]). From (5.14)–(5.17), we deduce that $u \in W^{0,1}(\mathbb{X}^{D_T})$ and $\sigma \nabla_{\mathbb{X}} u(\mathbf{X}^D) = Z^{s,x}$, $\mathrm{d}t \otimes P'_{s,x}$-a.e. Putting

$$A_t = u(\mathbf{X}_t) - u(s, x) - \int_0^t \sigma \nabla_{\mathbb{X}} u(\mathbf{X}_r)\,\mathrm{d}B_r$$

we see that A is a CAF of finite variation and $P'_{s,x}(R^{s,x}_t = A_t,\ t \in [0, \zeta_\tau]) = 1$, which proves the proposition in the case u is bounded. In the general case, (5.15) still holds. By the Itô–Tanaka formula, for every $k > 0$, we have

$$T_k(u)(\mathbf{X}_t) = T_k(u(s,x)) + \int_0^t \mathbf{1}_{(-k,k]}(u(\mathbf{X}_r))\, \mathrm{d}R_r^{s,x} + \int_0^t \mathbf{1}_{(-k,k]}(u(\mathbf{X}_r)) Z_r^{s,x}\, \mathrm{d}B_r + \frac{1}{2}(L_t^k - L_t^{-k}), \quad t \in [0, \zeta_\tau],$$

where L^k (respectively, L^{-k}) is the local time of the process $u(\mathbf{X})$ at k (respectively, $-k$). By the first step of the proof, $T_k(u) \in W^{0,1}(\mathbb{X}^{D_T})$ for every $k \geq 0$. Since u is quasi-continuous, $u \in FB$ and hence, by Lemma 5.5, $u \in W^{0,1}(\mathbb{X}^{D_T})$. By the first step of the proof,

$$Z^{s,x}\mathbf{1}_{(-k,k]}(u(\mathbf{X})) = \sigma\nabla_{\mathbb{X}}(T_k(u))(\mathbf{X}), \quad \mathrm{d}t \otimes P'_{s,x}\text{-a.e.}$$

Hence, by Lemma 5.4,

$$Z^{s,x}\mathbf{1}_{(-k,k]}(u(\mathbf{X})) = \mathbf{1}_{(-k,k]}(u(\mathbf{X}))\sigma\nabla_{\mathbb{X}}u(\mathbf{X}), \quad \mathrm{d}t \otimes P'_{s,x}\text{-a.e.}$$

for every $k \geq 0$, which implies that

$$Z^{s,x} = \sigma\nabla_{\mathbb{X}}u(\mathbf{X}), \quad \mathrm{d}t \otimes P'_{s,x}\text{-a.e.}$$

The rest of the proof runs as in the first step. □

THEOREM 5.12. *Assume* (H1)–(H4). *Then, there exists a unique solution u of system* (5.1). *Moreover, there exists a version of u (still denoted by u) such that* (5.7) *is satisfied for every $(s, x) \in F$, where F is defined in Corollary* 5.9.

Proof. By Proposition 5.9, for every $(s, x) \in F$, there exists a solution $(Y^{s,x}, Z^{s,x})$ of $\mathrm{BSDE}_{s,x}(\varphi, D, f + \mathrm{d}\mu)$ such that $(Y^{s,x}, Z^{s,x}) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and $Y^{s,x}$ is of class (D). By [19, Proposition 3.5], $u(\mathbf{X}_t) = Y_t^{s,x}$, $P'_{s,x}$-a.s. for every $(s, x) \in F$ and every $t \in [0, T_\tau]$, where $u(s, x) = E'_{s,x}Y_0^{s,x}$. Our aim is to show that u is quasi-càdlàg, belongs to $W^{0,1}(\mathbb{X}^{D_T})$ and representation (5.8) holds q.e. Note that we cannot apply directly Propositions 5.10, 5.11 because we do not know that $u(\mathbf{X})$ is continuous. To overcome the difficulty, let us consider a solution $(Y^{1,s,x}, Z^{1,s,x}) \in \mathcal{D}^q \otimes M^q$, $q \in (0, 1)$, of the $\mathrm{BSDE}_{s,x}(\varphi, D, \mu)$ such that $Y^{1,s,x}$ is of class (D). By [19, Proposition 3.7], $u_1(s, x) = E'_{s,x}Y_0^{1,s,x}$ is quasi-càdlàg, $u_1 \in \mathcal{T}_2^{0,1}$ and for every $(s, x) \in F$,

$$u_1(\mathbf{X}_t) = Y_t^{1,s,x}, \quad t \in [0, \zeta_\tau], \ P'_{s,x}\text{-a.s.}, \quad \sigma\nabla u_1(\mathbf{X}) = Z^{1,s,x}, \quad \mathrm{d}t \otimes P'_{s,x}\text{-a.e.} \tag{5.18}$$

Observe that

$$Y_t^{s,x} - Y_t^{1,s,x} = \int_t^{\zeta_\tau} f(\mathbf{X}_r, Y_r^{s,x})\, \mathrm{d}r + \int_t^{\zeta_\tau} (Z_r^{s,x} - Z_r^{1,s,x})\, \mathrm{d}B_r, \ t \in [0, \zeta_\tau], \ P'_{s,x}\text{-a.s.}$$

Applying Proposition 5.10 to each coordinate of the process $Y^{s,x} - Y^{1,s,x}$, we conclude that there is a quasi-continuous function $v : D_T \to \mathbb{R}^N$ such that for every $(s, x) \in F$,

$$Y_t^{s,x} - Y_t^{1,s,x} = v(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

From this and (5.18),

$$Y_t^{s,x} = u_1(\mathbf{X}_t) + v(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

But

$$v(s,x) = E'_{s,x} Y_0^{s,x} - E'_{s,x} Y_0^{1,s,x} = u_1(s,x) - u(s,x)$$

for q.e. $(s,x) \in D_T$. Therefore,

$$Y_t^{s,x} = u(\mathbf{X}_t), \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

It follows in particular that u is quasi-càdlàg. Since u_1 is quasi-càdlàg, it belongs to FB. Consequently, by Corollary 5.6, $u_1 \in W^{0,1}(\mathbb{X}^{D_T})$. Therefore, from (5.18) and Proposition 5.11 applied to each coordinate of the function v, it follows that $u \in W^{0,1}(\mathbb{X}^{D_T})$ and

$$\sigma \nabla_{\mathbb{X}} u(\mathbf{X}) = Z^{s,x}, \quad \mathrm{d}t \otimes P'_{s,x}\text{-a.e.}$$

for every $(s,x) \in F$. Thus, u is a solution of (5.1). Uniqueness follows from Theorem 3.6. □

PROPOSITION 5.13. *Let u be a solution of system (5.1). Then, $\nabla_{\mathbb{X}} u \in L^q_{loc}(D_T)$ for every $q \in (0,1)$.*

Proof. Since u is of class (FD), $u(\mathbf{X})$ is of class (D) on $[0, \zeta_\tau]$ under $P'_{s,x}$ for q.e. $(s,x) \in D_T$. Therefore, $(u(\mathbf{X}), \sigma\nabla_{\mathbb{X}} u(\mathbf{X}))$ is a unique solution of $\mathrm{BSDE}_{s,x}(\varphi, D, f + \mathrm{d}\mu)$. By [3, Proposition 6.4], $u \in \mathcal{D}^q$ and $\nabla_{\mathbb{X}} u \in M^q$ for $q \in (0,1)$. Applying the Itô–Meyer formula to (5.7) and using (H2) and the fact that u is of class (FD), we get

$$|u(\mathbf{X}_t)| \le E'_{s,x}\left(|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_r, 0)|\,\mathrm{d}r + \int_0^{\zeta_\tau} \mathrm{d}|A^\mu|_r \Big| \mathcal{F}'_t\right).$$

By [3, Lemma 6.1], for every $q \in (0,1)$,

$$E'_{s,x} \sup_{0\le t\le \zeta_\tau} |u(\mathbf{X}_t)|^q \le (1-q)^{-1} E'_{s,x}\Bigg(1 + |\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_r, 0)|\,\mathrm{d}r + \int_0^{\zeta_\tau} \mathrm{d}|A^\mu|_r\Bigg).$$

By the above estimate and Lemma 3.1,

$$E'_{s,x}\left(\int_0^{\zeta_\tau} |\sigma\nabla_{\mathbb{X}} u(\mathbf{X}_r)|^2\,\mathrm{d}r\right)^{q/2} \le c(q) E'_{s,x}\Bigg(1 + |\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_r, 0)|\,\mathrm{d}r + \int_0^{\zeta_\tau} \mathrm{d}|A^\mu|_r\Bigg)$$

for every $q \in (0,1)$. Since $q \in (0,1)$, we have

$$E'_{s,x}\left(\int_0^{\zeta_\tau} |\sigma\nabla_{\mathbb{X}} u(\mathbf{X}_r)|^2\,\mathrm{d}r\right)^{q/2} \geq \Lambda^{-1} E'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u(\mathbf{X}_r)|^q\,\mathrm{d}r \cdot \zeta_\tau^{q/2-1}\right)$$
$$\geq \Lambda^{-1} T^{q/2-1} E'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u(\mathbf{X}_r)|^q\,\mathrm{d}r\right).$$

By the above,

$$\int\!\!\int_{D_T} |\nabla_{\mathbb{X}} u(r,y)|^q p_D(s,x,r,y)\,\mathrm{d}r\,dy = E'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u(\mathbf{X}_r)|^q\,\mathrm{d}r\right) < \infty$$

for q.e. $(s,x) \in D_T$, where p_D is the transition density of the process $\mathbb{X}$ killed on exiting D. Therefore, the desired result follows from the well-known fact that $p_D(s,x,\cdot,\cdot)$ is continuous and strictly positive on $(s,T] \times D$ (see [1]). $\square$

REMARK 5.14. From [19], it follows that if u is a probabilistic solution of (5.1) such that $f(\cdot,u) \in L^1(D_T)$, then $u \in \mathcal{T}_2^{0,1}$, $u \in L^q(0,T;W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$ and u is a renormalized (entropy) solution of (5.1).

6. Large-time asymptotics

In this section, we prove our main results (1.10) and (1.11) on large-time behavior of solutions of (1.1) in case A, μ and f do not depend on time. A key role in obtaining (1.10), (1.11) play some a priori estimates on solutions of BSDEs which we present below.

6.1. Large-time estimates for the solutions of BSDEs

Let μ be a soft measure such that A^μ is a CAF of $\mathbb{X}'$. Let $\zeta_\tau^n = (n-\tau(0)) \wedge \zeta$ and let (Y^n, Z^n) be a solution of the BSDE$_{s,x}$

$$Y_t^n = \varphi(\mathbf{X}_{n-\tau(0)})\mathbf{1}_{\{\zeta > n-\tau(0)\}} + \int_t^{\zeta_\tau^n} f(\mathbf{X}_r, Y_r^n)\,\mathrm{d}r$$
$$+ \int_t^{\zeta_\tau^n} \mathrm{d}A_r^\mu - \int_t^{\zeta_\tau^n} Z_r^n\,\mathrm{d}B_r, \quad 0 \leq t \leq \zeta_\tau^n$$

such that $(Y^n, Z^n) \in \mathcal{D}^q \otimes M^q$ for $q \in (0,1)$ and Y^n is of class (D). In the remainder of the paper, we adopt the convention that $Y_t^n = 0$, $t \geq n-\tau(0)$ and $Z_r^n = 0$, $r \geq n-\tau(0)$.

PROPOSITION 6.1. *Assume that* (H1)–(H4) *are satisfied with* $\alpha \le 0$. *Let* $n < m$ *and* $\delta Y_t = Y^m_t - Y^n_t$, $\delta Z_t = Z^m_t - Z^n_t$. *Then, for every* $q \in (0,1)$,

$$
\begin{aligned}
&E'_{s,x} \sup_{t\ge 0} |\delta Y_t|^q + \epsilon E'_{s,x}\left(\int_0^\zeta |\delta Z_r|^2\,\mathrm{d}r\right)^{q/2}\\
&\quad \le (1-q)^{-1}(1+\epsilon 2C_q)E'_{s,x}\Bigg(|\varphi(\mathbf{X}_{n-s})|^q\mathbf{1}_{\{\zeta>n-s\}} + |\varphi(\mathbf{X}_{m-s})|^q\mathbf{1}_{\{\zeta>m-s\}}\\
&\qquad + \left(\int_n^\zeta \mathrm{d}|A^\mu|_r\right)^q + \left(\int_n^\zeta |f(\mathbf{X}_r,0)|\,\mathrm{d}r\right)^q\Bigg)
\end{aligned}
$$

for $\epsilon = 0, 1$, *where* C_q *is the constant from Lemma* 3.1.

Proof. Observe that

$$
Y^n_t = Y^n_0 - \int_0^t \mathbf{1}_{[0,\zeta^n_\tau]}(r) f(\mathbf{X}_r, Y^n_r)\,\mathrm{d}r - \int_0^t \mathbf{1}_{[0,\zeta^n_\tau]}(r)\,\mathrm{d}A^\mu_r + \int_0^t dV^n_r + \int_0^t Z^n_r\,\mathrm{d}B_r
$$

for $t \ge 0$, where

$$
V^n_t = \begin{cases} 0, & t < n - \tau(0),\\ -Y^n_{n-\tau(0)}, & t \ge n-\tau(0). \end{cases}
$$

Put

$$
\begin{aligned}
\psi(t) &= -Y^m_0 + \int_0^t \mathbf{1}_{[0,\zeta^m_\tau]}(r) f(\mathbf{X}_r, Y^m_r)\,\mathrm{d}r + \int_0^t \mathbf{1}_{[0,\zeta^m_\tau]}(r)\,\mathrm{d}A^\mu_r - \int_0^t dV^m_r\\
&\quad + Y^n_0 - \int_0^t \mathbf{1}_{[0,\zeta^n_\tau]}(r) f(\mathbf{X}_r, Y^n_r)\,\mathrm{d}r - \int_0^t \mathbf{1}_{[0,\zeta^n_\tau]}(r)\,\mathrm{d}A^\mu_r + \int_0^t dV^n_r, \quad t \ge 0.
\end{aligned}
$$

By the Itô–Meyer formula (see Corollary 2.3) and standard localization procedure for local martingale $N = \int_0^\cdot \langle \hat{\mathrm{sgn}}(\delta Y_{r-}), \delta Z_r\,\mathrm{d}B_r\rangle$, for $t \le m - \tau(0)$, we have

$$
|\delta Y_t| \le E'_{s,x}\left(|\delta Y_{m-s}| + \int_t^{m-s} \langle \hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}\psi(r)\rangle | \mathcal{F}'_t\right). \tag{6.1}
$$

Suppose that $n - \tau(0) < t \le m - \tau(0)$. Since $|\delta Y_{m-\tau(0)}| = 0$, it follows from (6.1) that

$$
\begin{aligned}
|\delta Y_t| &\le E'_{s,x}\Bigg(\int_t^{m-s} \langle \hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta^m_\tau]}(r),\ f(\mathbf{X}_r, Y^m_r)\rangle\,\mathrm{d}r\\
&\quad + \int_t^{m-s} \langle \mathbf{1}_{[0,\zeta^m_\tau]}(r)\hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}A^\mu_r\rangle - \int_t^{m-s} \langle \hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta^m_\tau]}(r), dV^m_r\rangle | \mathcal{F}'_t\Bigg).
\end{aligned}
$$

Moreover,

$$
\int_t^{\zeta^m_\tau} \langle \hat{\mathrm{sgn}}(\delta Y_{r-}),\ f(\mathbf{X}_r, Y^m_r)\rangle\mathrm{d}r = \int_t^{\zeta^m_\tau} \langle \hat{\mathrm{sgn}}(Y^m_{r-}),\ f(\mathbf{X}_r, Y^m_r)\rangle\mathrm{d}r \le \int_{n-\tau(0)}^\zeta |f(\mathbf{X}_r,0)|\,\mathrm{d}r
$$

and

$$\int_t^{\zeta_\tau^m}\langle \hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}A_r^\mu\rangle \le \int_{n-\tau(0)}^{\zeta}\mathrm{d}|A^\mu|_r,\quad \int_t^{\zeta_\tau^m}\langle \hat{\mathrm{sgn}}(\delta Y_{r-}),\ dV_r^m\rangle \le |\varphi(\mathbf{X}_{m-\tau(0)})|\mathbf{1}_{[0,\zeta_\tau^m]}.$$

By the above estimates, if $n-\tau(0)<t\le m-\tau(0)$, then

$$|\delta Y_t|\le E'_{s,x}\left(\int_{n-s}^{\zeta}\mathrm{d}|A^\mu|_r+\int_{n-s}^{\zeta}|f(\mathbf{X}_r,0)|\,\mathrm{d}r+|\varphi(\mathbf{X}_{m-s})|\mathbf{1}_{[0,\zeta_\tau^m]}|\mathcal{F}'_t\right)\equiv I(n,m).$$

Suppose now that $0\le t\le n-\tau(0)$. Then, by (6.1),

$$\begin{aligned}
|\delta Y_t| \le{}& E'_{s,x}\left(\int_t^{m-s}\langle \mathbf{1}_{[0,\zeta_\tau^m]}(r)f(\mathbf{X}_r,Y_r^m)-\mathbf{1}_{[0,\zeta_\tau^n]}(r)f(\mathbf{X}_r,Y_r^n),\hat{\mathrm{sgn}}(\delta Y_r\rangle\,\mathrm{d}r\right.\\
&+\int_t^{m-s}\langle(\mathbf{1}_{[0,\zeta_\tau^m]}(r)-\mathbf{1}_{[0,\zeta_\tau^n]}(r))\hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}A_r^\mu\rangle\\
&\left.+\int_t^{m-s}\langle\hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta_\tau^n]}(r),dV_r^n\rangle-\int_t^{m-s}\langle\hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta_\tau^m]}(r),dV_r^m\rangle|\mathcal{F}_t\right)\\
\le{}& I(n,m)+E'_{s,x}\left(\int_t^{n-s}\langle\mathbf{1}_{[0,\zeta_\tau^m]}(r)f(\mathbf{X}_r,Y_r^m)-\mathbf{1}_{[0,\zeta_\tau^n]}(r)f(\mathbf{X}_r,Y_r^n),\hat{\mathrm{sgn}}(\delta Y_r)\rangle\mathrm{d}r\right.\\
&+\int_t^{n-s}\langle(\mathbf{1}_{[0,\zeta_\tau^m]}(r)-\mathbf{1}_{[0,\zeta_\tau^n]}(r))\hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}A_r^\mu\rangle\\
&\left.+\int_t^{n-s}\langle\hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta_\tau^n]}(r),dV_r^n\rangle-\int_t^{n-s}\langle\hat{\mathrm{sgn}}(\delta Y_{r-})\mathbf{1}_{[0,\zeta_\tau^m]}(r),\ dV_r^m\rangle|\mathcal{F}'_t\right).
\end{aligned}$$

The last term is equal to zero. By (H2),

$$\begin{aligned}
&\int_t^{n-\tau(0)}\langle\mathbf{1}_{[0,\zeta_\tau^m]}(r)f(\mathbf{X}_r,Y_r^m)-\mathbf{1}_{[0,\zeta_\tau^n]}(r)f(\mathbf{X}_r,Y_r^n)),\hat{\mathrm{sgn}}(\delta Y_{r-})\rangle\,\mathrm{d}r\\
&\quad=\int_t^{n-\tau(0)}\langle\mathbf{1}_{[0,\zeta_\tau^n]}(r)(f(\mathbf{X}_r,Y_r^m)-f(\mathbf{X}_r,Y_r^n)),\hat{\mathrm{sgn}}(\delta Y_{r-})\rangle\,\mathrm{d}r\le 0.
\end{aligned}$$

Moreover,

$$\int_t^{n-\tau(0)}\langle(\mathbf{1}_{[0,\zeta_\tau^m]}(r)-\mathbf{1}_{[0,\zeta_\tau^n]}(r))\hat{\mathrm{sgn}}(\delta Y_{r-}),\ \mathrm{d}A_r^\mu\rangle=0.$$

By what has been already proved, for every $t\ge 0$,

$$\begin{aligned}
|\delta Y_t|\le{}& E'_{s,x}\left(|\varphi(\mathbf{X}_{n-s})|\mathbf{1}_{\{\zeta>n-s\}}+|\varphi(\mathbf{X}_{m-s})|\mathbf{1}_{\{\zeta>m-s\}}\right.\\
&\left.+\int_{n-s}^{\zeta}\mathrm{d}|A^\mu|_r+\int_{n-s}^{\zeta}|f(\mathbf{X}_r,0)|\,\mathrm{d}r|\mathcal{F}'_t\right).
\end{aligned}\tag{6.2}$$

The desired estimate now follows from Lemma 6.1 in [3] and Lemma 3.1. □

6.2. Large-time behavior of solutions of PDEs

In this subsection, we consider the operator A defined by (4.10) (with coefficients not depending on time). Let us recall that for a nonnegative Borel measure μ on D_T (respectively, D), we write

$$R^{0,T}\mu(s,x)=\int_{D_T}p_D(s,x,t,y)\,\mathrm{d}\mu(t,y),\quad \text{(resp.}\quad R\mu(x)=\int_D G_D(x,y)\,\mathrm{d}\mu(y)).$$

LEMMA 6.2. *Let $\mu\in\mathcal{M}^+_{0,b}(D_T)$. Then, there exists a PAF A^μ of $\mathbf{X}$ such that for every $(s,x)\in D_T$, if $R^{0,T}\mu(s,x)<\infty$, then*

$$E'_{s,x}\int_0^{\zeta_\tau}\eta(\mathbf{X}_t)\,\mathrm{d}A^\mu_t=\int_{D_T}\eta(\theta,y)p_D(s,x,t,y)\,\mathrm{d}\mu(t,y)$$

for every bounded $\eta\in\mathcal{B}(D_T)$.

Proof. See [13]. □

We say that a Borel measure μ on D_T does not depend on time if for every $A\in\mathcal{B}([0,T])$ and $B\in\mathcal{B}(D)$,

$$\mu(A\times B)=\lambda(A)\cdot\tilde{\mu}(B)\tag{6.3}$$

for some Borel measure $\tilde{\mu}$ on D (Here, λ stands for the Lebesgue measure). Since $\tilde{\mu}(B)=\mu([0,1]\times B)$ for $B\in\mathcal{B}(D)$, $\tilde{\mu}$ is uniquely determined by μ. From now on, given μ not depending on time, we will denote by $\tilde{\mu}$ the Borel measure on D determined by (6.3). It is clear that if μ is soft with respect to cap, then $\tilde{\mu}$ is soft with respect to cap $_N$ (see Sect. 4.3). It is known (see [9]) that there exists a unique continuous additive functional $A^{\tilde{\mu}}$ of $\mathbb{X}=\{(X,P_x\equiv P'_{0,x}):x\in\mathbb{R}^d\}$ in the Revuz correspondence with measure $\tilde{\mu}$ or, equivalently, satisfying (4.12).

LEMMA 6.3. *Let $\mu\in\mathcal{M}^+_{0,b}(D_T)$ do not depend on time. Then, A^μ is continuous, for every $s\in[0,T]$, $R^{0,T}\mu(s,x)<\infty$ for q.e. $x\in D$ and*

$$A^\mu_t(0,\omega)=A^{\tilde{\mu}}_t(\omega)\quad \textit{for } P_x\text{-a.e.}\quad \omega\in\Omega.$$

Proof. Since (X,P_x) is time homogeneous, $p_D(s,x,t,y)=p_D(t-s,x,y)$ for $t>s$, $x,y\in D$. Taking into account that $\mu=\mathrm{d}t\otimes\tilde{\mu}$, we therefore have

$$\begin{aligned}\int_{D_T}p_D(s,x,\theta,y)\,\mathrm{d}\mu(\theta,y)&=\int_0^T\int_D p_D(\theta-s,x,y)\,\mathrm{d}\theta\,\tilde{\mu}(dy)\\&\le\int_0^\infty\int_D p_D(t,x,y)\tilde{\mu}(dy)\,\mathrm{d}t\\&=E_x\int_0^\zeta \mathrm{d}A^{\tilde{\mu}}_t.\end{aligned}$$

It is known (see [9]) that for every bounded smooth measure $\tilde{\mu}$ on D, the last integral above is finite for q.e. $x \in D$. This proves that for every $s \in [0, T]$, $R^{0,T}\mu(s, x) < \infty$ for q.e. $x \in D$. By [13], $E'_{s,x}(A^{\mu}_t - A^{\mu}_{t-}) = \int_D p_D(s, x, t, y)\mu(t, dy)$, where $\mu(t, B) = \mu(\{t\} \times B)$ for $B \in \mathcal{B}(D)$. Since $\mu = \lambda \otimes \tilde{\mu}$, we conclude that $\mu(t, dy) \equiv 0$ for every $t \in [0, T]$, which implies that A^{μ} is continuous. We may assume that $\tilde{\mu} \in H^{-1}(D)$. If not, we argue as follows. It is known (see [9]) that for every smooth measure $\tilde{\mu}$ on D, there exists a generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot \tilde{\mu} \in H^{-1}(D)$ for every $n \geq 1$. We prove the assertion for each measure $\mathbf{1}_{F_n} \cdot \tilde{\mu}$, and then, it is a simple matter to deduce the assertion for $\tilde{\mu}$. By the construction of the AF (see [13]), for every $(s, x) \in D_T$ such that $R^{0,T}\mu(s, x) < \infty$,

$$E'_{s,x} \sup_{0 \leq t \leq T_\tau} |A^{\mu_n}_t - A^{\mu}_t|^2 \to 0, \tag{6.4}$$

where $\mu_n = f_n \cdot m_1$, $f_n = n(u - u_n)$, $A^{\mu_n}_t(\omega') = \int_0^t f_n(\mathbf{X}_r(\omega'))\,\mathrm{d}r$, u is a solution to the problem

$$\begin{cases} \frac{\partial u}{\partial t} - Au = \mu, \\ u(0, \cdot) = \varphi, \quad u(t, \cdot)_{|\partial D} = 0, \ t \in (0, T] \end{cases}$$

with some $\varphi \in L^2(D)$ and u_n, $n \in \mathbb{N}$, are solutions to the problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - Au_n = n(u - u_n), \\ u_n(0, \cdot) = \varphi_n, \quad u_n(t, \cdot)_{|\partial D} = 0, \ t \in (0, T] \end{cases}$$

with $\varphi_n \in L^2(D)$ such that $\varphi_n \to u(0, \cdot)$ in $L^2(D)$. Since μ does not depend on time, one can choose $u \in H^1_0(D)$ such that $-Au = \tilde{\mu}$ and $u_n \in H^1_0(D)$ such that $-Au_n = n(u - u_n)$. By the construction of the PCAF $A^{\tilde{\mu}}$ of (X, P_x),

$$\int_0^t f_n(X_r(\omega))\,\mathrm{d}r \to A^{\tilde{\mu}}_t, \quad t \geq 0$$

for $P_{s,x}$-a.e. $\omega \in \Omega$. But the sequence $\{\int_0^t f_n(X_r(\omega))\,\mathrm{d}r\}$ is convergent for P_x-a.e. $\omega \in \Omega$ iff $\{\int_0^t f_n(\mathbf{X}_r(\omega'))\,\mathrm{d}r\}$ is convergent for $P'_{0,x}$-a.e. $\omega' \in \Omega'$, which in view of (6.4) gives the desired result. □

From now on, we assume that the measure μ and the right-hand side f do not depend on time. Let us consider the following parabolic system of PDEs

$$\begin{cases} \frac{\partial u^k}{\partial t} - Au^k = f^k(x, u) + \mu^k, \quad k = 1, \ldots, N, \\ u_{|\partial D}(t, \cdot) = 0, \ t \in (0, T], \qquad u(0, \cdot) = \varphi \end{cases} \tag{6.5}$$

and elliptic system of PDEs

$$\begin{cases} -Av^k = f^k(x, v) + \tilde{\mu}^k \quad \text{in } D, \quad k = 1, \ldots, N, \\ v_{|\partial D} = 0. \end{cases} \tag{6.6}$$

Let $F_0 = \{x \in D; (0, x) \in F\}$, where F is the set defined in Proposition 5.9 with $T = \infty$ (see (1.12)).

THEOREM 6.4. *Assume* (H1)–(H4) *with* $\alpha \le 0$. *Let* u, v *be solutions of* (6.5) *and* (6.6)*, respectively. Then, for every* $t > 0$, $q \in (0,1)$ *and every* $x \in F_0$,

$$|u(t,x) - v(x)| \le c(\Lambda)(1-q)^{-1/q}\left(|P_t\varphi(x)|P_x^{(1-q)/q}(\zeta^0 > t) + |(P_tR\tilde{\mu})(x)| + |(P_tRf(\cdot,0))(x)|\right). \tag{6.7}$$

Proof. By Proposition 5.9, for every $x \in F_0$ (hence for q.e. $x \in D$ by Lemma 6.3), there exists a solution (Y^n, Z^n) of $\mathrm{BDSE}_{0,x}$

$$Y_t^n = \varphi(\mathbf{X}_{n-\tau(0)})\mathbf{1}_{\{\zeta > n-\tau(0)\}} + \int_t^{\zeta_\tau^n} f(\mathbf{X}_r, Y_r^n)\,\mathrm{d}r + \int_t^{\zeta_\tau^n} \mathrm{d}A_r^\mu - \int_t^{\zeta_\tau^n} Z_r^n\,\mathrm{d}B_r, \quad 0 \le t \le \zeta_\tau^n. \tag{6.8}$$

Let us recall that we put $Z_t^n = Y_t^n = 0$ for $t \ge n - \tau(0)$. By Proposition 6.1, for every $n < m$ and $x \in F_0$, we have

$$\begin{aligned} &E'_{0,x}\sup_{t\ge 0}|Y_t^n - Y_t^m|^q + E'_{0,x}\left(\int_0^\zeta |Z_r^n - Z_r^m|^2\,\mathrm{d}r\right)^{q/2} \\ &\le (1-q)^{-1}(1+2C_q)E'_{0,x}\Big(|\varphi(\mathbf{X}_n)|^q\mathbf{1}_{\{\zeta>n\}} + |\varphi(\mathbf{X}_m)|^q\mathbf{1}_{\{\zeta>m\}} \\ &\quad + \left(\int_n^\zeta \mathrm{d}|A^\mu|_r\right)^q + \left(\int_n^\zeta |f(\mathbf{X}_r,0)|\,\mathrm{d}r\right)^q\Big). \end{aligned} \tag{6.9}$$

For every $t > 0$,

$$\begin{aligned} E'_{0,x}|\varphi(\mathbf{X}_t)|^q\mathbf{1}_{\{\zeta>t\}} &= E_x|\varphi(X_t)|^q\mathbf{1}_{\{\zeta^0>t\}} \\ &\le (E_x|\varphi(X_t)|)^q \cdot P_x^{1-q}(\zeta^0 > t) \\ &= |(P_t\varphi(x)|^q \cdot P_x^{1-q}(\zeta^0 > t) \\ &\le c(\Lambda)t^{-qd/2}\|\varphi\|_{L^1}^q P_x^{1-q}(\zeta^0 > t). \end{aligned} \tag{6.10}$$

Also,

$$\begin{aligned} E'_{0,x}\left(\int_t^\zeta \mathrm{d}|A^\mu|_r\right)^q &\le (E'_{0,x}\int_t^\zeta \mathrm{d}|A^\mu|_r)^q = \left(E_x\int_t^{\zeta^0}\mathrm{d}|A^{\tilde{\mu}}|_r\right)^q = \left(E_xE_{X_t}\int_0^{\zeta^0}\mathrm{d}|A^{\tilde{\mu}}|_r\right)^q \\ &= (E_x(R\tilde{\mu})(X_t))^q = (P_t(R\tilde{\mu})(x))^q \le c(\Lambda)t^{-dq/2}\|R\tilde{\mu}\|_{L^1}^q. \end{aligned} \tag{6.11}$$

Since $\|R\tilde{\mu}\|_{L^1} = \langle R\tilde{\mu}, 1\rangle_{L^2}$ and by (4.13), $(R1)(x) = E_x\zeta^0 \le c(d,\Lambda)|D|^{d/2}$, it follows that

$$\|R\tilde{\mu}\|_{L^1} \le c(d,\Lambda)|D|^{d/2}\|\tilde{\mu}\|_{TV}. \tag{6.12}$$

Similarly, since $\int_0^t |f(\mathbf{X}_r,0)|\,\mathrm{d}r = \int_0^t \mathrm{d}|A^\nu|_r,\ t \ge 0$, where $\nu = f(\cdot,0)\cdot m$, we have

$$E'_{0,x}\left(\int_t^\zeta |f(\mathbf{X}_r,0)|\,\mathrm{d}r\right)^q \le c(d,\Lambda)t^{-dq/2}\|R\nu\|_{L^1}^q \tag{6.13}$$

and

$$\|R\nu\|_{L^1} \le c(d,\Lambda)|D|^{d/2}\|f(\cdot,0)\|_{L^1}. \tag{6.14}$$

From (6.9)–(6.14), it follows that

$$E'_{0,x}\sup_{t\ge 0}|Y^n_t - Y^m_t|^q + E'_{0,x}\left(\int_0^\zeta |Z^n_r - Z^m_r|^2\,\mathrm{d}r\right)^{q/2} \to 0 \tag{6.15}$$

as $n, m \to \infty$. Let us denote by (Y, Z) the limit of the sequence $\{(Y^n, Z^n)\}$. It is clear from the above that $(Y, Z) \in S^q \otimes M^q$ for $q \in (0, 1)$. For fixed $\varepsilon, R > 0$, we have

$$\begin{aligned}
&P'_{0,x}\left(\int_0^\zeta |f(\mathbf{X}_r, Y^n_r) - f(\mathbf{X}_r, Y_r)|\,\mathrm{d}r > \varepsilon\right)\\
&\quad\le P'_{0,x}\left(\int_0^\zeta |f(\mathbf{X}_r, Y^n_r) - f(\mathbf{X}_r, Y_r)|\,\mathrm{d}r > \varepsilon, \sup_{t\ge0}|Y_t| \le R, \sup_{t\ge0}|Y^n_t - Y_t| \le R\right)\\
&\qquad + P'_{0,x}(\sup_{t\ge0}|Y_t| > R) + P'_{0,x}(\sup_{t\ge0}|Y^n_t - Y_t| > R)\\
&\quad\equiv I_1(n, R, \varepsilon) + I_2(R) + I_3(n, R).
\end{aligned}$$

By (H1), (H4), and the Lebesgue dominated convergence theorem, $I_1(n, R, \varepsilon) \to 0$ as $n \to \infty$. By (6.9), $I_3(n, R) \to 0$ as $n \to \infty$. Finally, since $Y \in S^q$ for $q \in (0, 1)$, $I_2(R) \to 0$ as $R \to +\infty$. This shows that

$$\int_t^\zeta f(\mathbf{X}_r, Y^n_r)\,\mathrm{d}r \to \int_t^\zeta f(\mathbf{X}_r, Y_r)\,\mathrm{d}r \tag{6.16}$$

in probability $P'_{0,x}$ uniformly on $[0, \zeta]$. By (6.10), (6.15), and (6.16), we may pass to the limit in (6.8). We then get

$$Y_t = \int_t^\zeta f(\mathbf{X}_r, Y_r)\,\mathrm{d}r + \int_t^\zeta \mathrm{d}A^\mu_r - \int_t^\zeta Z_r\,\mathrm{d}B_r, \quad 0 \le t \le \zeta, \quad P'_{0,x}\text{-a.s.} \tag{6.17}$$

By the Itô–Meyer formula,

$$\begin{aligned}
|Y^n_t| &\le E'_{0,x}\left(\int_t^\zeta f(\mathbf{X}_r, Y^n_r)\cdot \hat{\operatorname{sgn}}(Y^n_r) + |\varphi(\mathbf{X}_n)|\mathbf{1}_{\{\zeta>n\}}|\mathcal{F}'_t\right)\\
&\le E'_{0,x}\left(\int_t^\zeta |f(\mathbf{X}_r, 0)|\,\mathrm{d}r|\mathcal{F}'_t\right) + E'_{0,x}\left(|\varphi(\mathbf{X}_n)|\mathbf{1}_{\{\zeta>n\}}|\mathcal{F}'_t\right)\\
&= E_x\left(\int_t^{\zeta^0} |f(X_r, 0)|\,\mathrm{d}r|\mathcal{F}_t\right) + E_x(|\varphi(X_n)|\mathbf{1}_{\{\zeta^0>n\}}|\mathcal{F}_t).
\end{aligned} \tag{6.18}$$

But for $t < n$,

$$E_x\left(|\varphi(X_n)||\mathcal{F}_t\right) = E_{X_t}|\varphi(X_{n-t})| = (P_{n-t}|\varphi|)(X_t) \le c(\Lambda)(n-t)^{-d/2}\|\varphi\|_{L^1}.$$

Therefore, letting $n \to \infty$ in (6.18), we obtain

$$|Y_t| \le E_x\left(\int_0^\zeta |f(X_r, 0)|\,\mathrm{d}r|\mathcal{F}_t\right).$$

From the above inequality, we conclude that Y is of class (D). Set $(Y^{0,n}(\omega), Z^{0,n}(\omega)) = (Y^n(0,\omega), Z^n(0,\omega))$ and $(Y^0(\omega), Z^0(\omega)) = (Y(0,\omega), Z(0,\omega))$ for $w \in \Omega$. Then, from (6.15), (6.17) and the fact that

$$E_x \sup_{t\geq 0} |Y_t^{0,n} - Y_t^0|^q + E_x \left(\int_0^{\zeta^0} |Z_r^{0,n} - Z_r^0|^2 \, \mathrm{d}r \right)^{q/2} \to 0 \tag{6.19}$$

it follows that

$$Y_t^0 = \int_t^{\zeta^0} f(X_r, Y_r^0) \, \mathrm{d}r + \int_t^{\zeta^0} \mathrm{d}A_r^{\tilde{\mu}} - \int_t^{\zeta^0} Z_r^0 \, \mathrm{d}B_r^0, \quad 0 \leq t \leq \zeta^0, \quad P_x\text{-a.s.}$$

By what has already been proved, the pair (Y^0, Z^0) has integrability properties under which the solution to $\mathrm{BSDE}_x(\zeta, f + \mathrm{d}\tilde{\mu})$ is unique (see [18]). Therefore, from [15], it follows that (Y^0, Z^0) has the representation

$$Y_t^0 = v(X_t), \quad 0 \leq t \leq \zeta^0, \ P_x\text{-a.s.}, \quad Z^0 = \sigma \nabla v(X) \text{ on } [0, \zeta^0] \times \Omega, \quad \mathrm{d}t \otimes P_x\text{-a.e.} \tag{6.20}$$

Moreover, by Theorem 5.12,

$$Y_t^n = u_n(\mathbf{X}_t), \ 0 \leq t \leq \zeta_\tau^n, \ P'_{0,x}\text{-a.s.}, \ Z^n = \sigma \nabla u_n(\mathbf{X}) \text{ on } [0, \zeta_\tau^n] \times \Omega', \ \mathrm{d}t \otimes P'_{0,x}\text{-a.e.},$$

where u_n is a solution of the system

$$\begin{cases} \frac{\partial u_n}{\partial t} + A u_n = -f(x, u_n) - \mu, \\ u_n(n, \cdot) = \varphi, \quad u_n(t, \cdot)_{|\partial D} = 0, \quad t \in [0, n) \end{cases}$$

Therefore, if we put $\zeta_\tau^{n,0}(\omega) = \zeta_\tau^n(0, \omega)$, then

$$Y_t^{0,n} = u_n(t, X_t), \quad 0 \leq t \leq \zeta_\tau^{n,0}, \ P_x\text{-a.s.} \tag{6.21}$$

and

$$Z^{0,n} = \sigma \nabla u_n(\cdot, X) \quad \text{on } [0, \zeta_\tau^{n,0}] \times \Omega, \ \mathrm{d}t \otimes P_x\text{-a.e.} \tag{6.22}$$

It is an elementary check that

$$u_n(t, x) = u(n - t, x), \quad t \in [0, n], \quad x \in D. \tag{6.23}$$

Letting $m \to \infty$ in (6.9) and using (6.19), we obtain

$$E_x \sup_{t\geq 0} |Y_t^{0,n} - Y_t^0|^q \leq (1-q)^{-1} E_x \left(|\varphi(X_n)|^q \mathbf{1}_{\{\zeta^0 > n\}} + \left(\int_n^{\zeta^0} \mathrm{d}|A^{\tilde{\mu}}|_r \right)^q + \left(\int_n^{\zeta^0} |f(X_r, 0)| \, \mathrm{d}r \right)^q \right).$$

From this and (6.10), (6.11), (6.20)–(6.23), we get (6.7). □

COROLLARY 6.5. *Assume* (H1)–(H4) *with* $\alpha \leq 0$. *Let* u *and* v *be solutions of* (6.5) *and (6.6), respectively. Then,* (1.11) *is satisfied for q.e.* $x \in D$. *In particular, for q.e.* $x \in D$, $u(t, x) \to v(x)$ *as* $t \to \infty$.

Proof. Follows from (6.7), (6.10)–(6.12), and (4.14). □

Acknowledgements

Research supported by Polish NCN Grant No. 2012/07/D/ST1/02107.

REFERENCES

[1] D.G. ARONSON, Non-Negative Solutions of Linear Parabolic Equations. *Ann. Sc. Norm. Super. Pisa* **22** (1968) 607–693.

[2] P. BÉNILAN, L. BOCCARDO, T. GALLOUËT, R. GARIEPY, M. PIERRE, J.- L. VAZQUEZ, L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995) 241–273.

[3] PH. BRIAND, B. DELYON, Y. HU, E. PARDOUX, L. STOICA, L^p solutions of Backward Stochastic Differential Equations. *Stochastic Process. Appl.* **108** (2003) 109–129.

[4] PH. BRIAND, R. CARMONA, BSDEs with polynomial growth generators. *J. Appl. Math. Stochastic Anal.* **13** (2000) 207–238.

[5] K.L. CHUNG, Z. ZHAO, From Brownian Motion to Schrödinger's Equation. Springer, Berlin Heidelberg, 1995.

[6] J. DRONIOU, A. PORRETTA, A. PRIGNET, Parabolic Capacity and Soft Measures for Nonlinear Equations. *Potential Anal.* **19** (2003) 99–161.

[7] N. EL KAROUI, C. KAPOUDJIAN, E. PARDOUX, S. PENG, M.C. QUENEZ, Reflected solutions of backward SDEs, and related obstacle problems for PDE's. *Ann. Probab.* **25** (1997) 702–737.

[8] A. FRIEDMAN, Partial differential equations of parabolic type. Prentice–Hall, Englewood Cliffs, N.J., 1964.

[9] M. FUKUSHIMA, Y. OSHIMA, M. TAKEDA, Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter, Berlin, 1994.

[10] S. HAMADÈNE, M. HASSANI, BSDEs with two reflecting barriers: the general result. *Probab. Theory Relat. Fields* **132** (2005) 237–264.

[11] L.V. KANTOROWICH, B.Z. VULIH, A.G. PINSKER, Functional Analysis in Partially Ordered Spaces (in Russian). Gostekhizdat, Moscow, 1950.

[12] J. KISYŃSKI, Convergence du type L. *Colloq. Math.* **7** (1960) 205–211.

[13] T. KLIMSIAK, Reflected BSDEs and the obstacle problem for semilinear PDEs in divergence form. *Stochastic Process. Appl.* **122** (2012) 134–169.

[14] T. KLIMSIAK, Cauchy problem for semilinear parabolic equation with time-dependent obstacles: a BSDEs approach. *Potential Anal.* **39** (2013) 99–140.

[15] T. KLIMSIAK, Semilinear elliptic systems with measure data. *Ann. Mat. Pura Appl. (4)* (2013) doi:10.1007/s10231-013-0364-4.

[16] T. KLIMSIAK, On Time-Dependent Functionals of Diffusions Corresponding to Divergence Form Operators. *J. Theoret. Probab.* **26** (2013) 437–473

[17] T. KLIMSIAK, Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations (2014) arXiv:1401.3643v1

[18] T. KLIMSIAK, A. ROZKOSZ, Dirichlet forms and semilinear elliptic equations with measure data. *J. Funct. Anal.* **265** (2013) 890–925.

[19] T. KLIMSIAK, A. ROZKOSZ, Obstacle problem for semilinear parabolic equations with measure data (2012). arXiv:1301.5795

[20] A. LEJAY, A probabilistic representation of the solution of some quasi-linear PDE with a divergence form operator. Application to existence of weak solutions of FBSDE, *Stochastic Process. Appl.* **110** (2004) 145–176.

[21] T. LEONORI, F. PETITTA, Asymptotic behavior for solutions of parabolic equations with natural growth terms and irregular data. *Asymptot. Anal.* **48** (2006), 219–233.

[22] P.- A. MEYER, Fonctionnelles multiplicatives et additives de Markov. *Ann. Inst. Fourier* **12** (1962) 125–230.

[23] L. ORSINA, A.C. PONCE, Semilinear elliptic equations and systems with diffuse measures. *J. Evol. Equ.* **8** (2008) 781–812.

[24] Y. OSHIMA, On construction of Markov processes associated with time dependent Dirichlet spaces. *Forum Math.* **4** (1992) 395–415.

[25] Y. OSHIMA, Time-dependent Dirichlet forms and related stochastic calculus. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** (2004) 281–316.

[26] E. PARDOUX, BSDEs, weak convergence and homogenization of semilinear PDEs, Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998). Kluwer Academic Publishers, Dordrecht, 1999, pp. 503–549.

[27] F. PETITTA, Asymptotic behavior of solutions for linear parabolic equations with general measure data. *C. R. Math. Acad. Sci. Paris* **344** (2007) 571-576.

[28] F. PETITTA, Asymptotic behavior of solutions for parabolic operators of Leray-Lions type and measure data. *Adv. Differential Equations* **12** (2007) 867-891.

[29] F. PETITTA, Large time behavior for solutions of nonlinear parabolic problems with sign-changing measure data. *Electron. J. Differential Equations* (2008) No. 132, 10 pp.

[30] D. REVUZ, Mesures associees aux fonctionnelles additives de Markov I. *Trans. Amer. Math. Soc.* **148** (1970) 501–531.

[31] A. ROZKOSZ, Weak convergence of diffusions corresponding to divergence form operators. *Stochastics Stochastics Rep.* **57** (1996) 129–157.

[32] A. ROZKOSZ, Backward SDEs and Cauchy problem for semilinear equations in divergence form. *Probab. Theory Related Fields* **125** (2003) 393–407.

[33] L. SŁOMIŃSKI, Stability of stochastic differential equations driven by general semimartingales. *Dissertationes Math.* **349** (1996) 113 pp.

[34] G. STAMPACCHIA, Équations elliptiques du second ordre à coefficients discontinus. *Séminaire de Mathémtiques Supérieures* **16** (1966) 326 pp.

T. Klimsiak
Faculty of Mathematics and
Computer Science,
Nicolaus Copernicus University,
Chopina 12/18, 87-100 Toruń,
Poland
E-mail: tomas@mat.umk.pl